ON RICCI COEFFICIENTS OF NULL HYPERSURFACES WITH TIME FOLIATION IN EINSTEIN VACUUM SPACE-TIME

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Abstract. The main objective of this paper is to control the geometry of null cones with time foliation in Einstein vacuum spacetime under the assumptions of small curvature flux and a weaker condition on the deformation tensor for $\mathbf{T}$. We establish a series of estimates on Ricci coefficients, which plays a crucial role to prove the improved breakdown criterion in [12].

1. Introduction

Consider a (3+1)-dimensional Einstein vacuum spacetime $(\mathbf{M}, \mathbf{g})$ foliated by Σ_t which are level hypersurfaces of a time function t monotonically increasing towards the future. Let $\mathbf{D}$ and ∇ denote the covariant differentiations with respect to $\mathbf{g}$ and the induced metric g on Σ_t respectively. We define on each Σ_t the lapse function n and the second fundamental form k by

$$n := (-\mathbf{g}(\mathbf{D}t, \mathbf{D}t))^{1/2} \quad \text{and} \quad k(X, Y) := -\mathbf{g}(\mathbf{D}_X \mathbf{T}, Y),$$

where $\mathbf{T}$ denotes the future directed unit normal to Σ_t and $X, Y \in T\Sigma_t$. For any coordinate chart $\mathcal{O} \subset \Sigma_{t_0}$ with coordinates $x = (x^1, x^2, x^3)$, let $x^0 = t, x^1, x^2, x^3$ be the transported coordinates obtained by following the integral curves of $\mathbf{T}$. Under these coordinates the metric $\mathbf{g}$ takes the form

$$\mathbf{g} = -n^2 dt^2 + g_{ij} dx^i dx^j, \quad \partial_t g_{ij} = -2n k_{ij}. \tag{1.1}$$

1.1. **Main result.** Consider an outgoing null cone contained in $(\mathbf{M}, \mathbf{g})$, whose vertex is denoted by p and intersections with Σ_t are denoted by S_t. The null vector l_ω in $T_p\mathbf{M}$ parametrized with $\omega \in \mathbb{S}^2$, is normalized by $\mathbf{g}(l_\omega, \mathbf{T}_p) = -1$. We denote by $\Gamma_\omega(s)$ the outgoing null geodesic from p with

$$\Gamma_\omega(0) = p, \quad \frac{d}{ds}\Gamma_\omega(0) = l_\omega$$

and define the null vector field L by

$$L(\Gamma_\omega(s)) = \frac{d}{ds}\Gamma_\omega(s).$$

Then $\mathbf{D}_L L = 0$. The affine parameter s of null geodesic is chosen such that $s(p) = 0$ and $L(s) = 1$. Let $\mathcal{H} = \cup_{0<t\le 1} S_t$, $t(p) = 0$[1] and suppose the exponential map $\mathcal{G}_t : \omega \to \Gamma_\omega(s(t))$ is a global diffeomorphism from $\mathbb{S}^2$ to S_t for any $t \in (0, 1]$. We now define a conjugate null vector $\underline{L}$ on $\mathcal{H}$ with $\mathbf{g}(L, \underline{L}) = -2$ and such that

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[1]we can always suppose $t(p) = 0$ and $0 \le t \le 1$ on a null cone by a standard rescaling of the coordinates (t, x) of $(\mathbf{M}, \mathbf{g})$. The normalized time function, for simplicity, is still denoted by t.

$\underline{L}$ is orthogonal to the leafs S_t. In addition we can choose an orthonormal frame $(e_A)_{A=1,2}$ tangent to S_t such that $(e_A)_{A=1,2}$, $e_3 = \underline{L}$, $e_4 = L$ form a null frame, i.e.

$$\mathbf{g}(L, \underline{L}) = -2,\ \mathbf{g}(L, L) = \mathbf{g}(\underline{L}, \underline{L}) = \mathbf{g}(L, e_A) = \mathbf{g}(\underline{L}, e_A) = 0,\ \mathbf{g}(e_A, e_B) = \delta_{AB}.$$

Let $a^{-1} = -\langle L, \mathbf{T}\rangle$ with $a(p) = 1$. It follows that along any null geodesic Γ_ω, there holds

$$\frac{dt}{ds} = n^{-1}a^{-1}, \quad t(p) = 0. \tag{1.2}$$

Let N be the outward unit normal of S_t on Σ_t. Then

$$\mathbf{T} = \frac{1}{2}\left(aL + a^{-1}\underline{L}\right), \qquad N = \frac{1}{2}\left(aL - a^{-1}\underline{L}\right). \tag{1.3}$$

We define the Ricci coefficients $\chi, \underline{\chi}, \zeta, \underline{\zeta}, \varpi$ via the frame equations

$$\begin{aligned} \mathbf{D}_A L &= \chi_{AB} e_B - \zeta_A L, &\qquad \mathbf{D}_A \underline{L} &= \underline{\chi}_{AB} e_B + \zeta_A \underline{L}, \\ \mathbf{D}_L \underline{L} &= 2\underline{\zeta}_A e_A, &\qquad \mathbf{D}_{\underline{L}} L &= 2\zeta_A e_A - 2\varpi L. \end{aligned}$$

Thus we also have

$$\mathbf{D}_L e_A = \nabla\!\!\!/_L e_A + \underline{\zeta}_A L, \quad \mathbf{D}_B e_A = \nabla\!\!\!/_B e_A + \frac{1}{2}\chi_{AB} e_3 + \frac{1}{2}\underline{\chi}_{AB} e_4$$

where $\nabla\!\!\!/$ denotes the covariant derivative restricted on S_t.

Let $\lambda = -\frac{1}{3}\mathrm{Tr}k$, where $\mathrm{Tr}k = g^{ij}k_{ij}$. We decompose $\hat{k}_{ij} := k_{ij} + \lambda g_{ij}$, the traceless part of k, relative to the orthonormal frame $\{N, e_A, A = 1, 2,\}$ along null cone $\mathcal{H}$ by introducing the following components

$$\eta_{AB} = \hat{k}_{AB} \qquad \epsilon_A = \hat{k}_{AN} \qquad \delta = \hat{k}_{NN}. \tag{1.4}$$

Denote by $\hat{\eta}_{AB}$ the traceless part of η. Since $\delta^{AB}\eta_{AB} = -\delta$, it is easy to see

$$\hat{\eta}_{AB} = \eta_{AB} + \frac{1}{2}\delta_{AB}\delta.$$

We denote by $\pi\!\!\!/$ one of the following S_t tangent tensors $\{\hat{\eta}, \delta, \epsilon, \lambda, -\nabla\!\!\!/ \log n, -\nabla_N \log n\}$. It is easy to check by definition that the Ricci coefficients $\zeta, \underline{\zeta}, \nu$ verify

$$\nu := -L(a) = -\nabla\!\!\!/_N \log n + \delta - \lambda, \tag{1.5}$$

$$\zeta_A = \nabla\!\!\!/_A \log a + \epsilon_A, \quad \underline{\zeta}_A = \nabla\!\!\!/_A \log n - \epsilon_A. \tag{1.6}$$

Let us define $\theta_{AB} := \langle \mathbf{D}_A N, e_B\rangle$. By definition of $\chi, \underline{\chi}$ and (1.3), it follows that

$$a\chi_{AB} = \theta_{AB} - k_{AB}, \quad a^{-1}\underline{\chi}_{AB} = -\theta_{AB} - k_{AB}, \tag{1.7}$$

$$a\mathrm{tr}\chi = \mathrm{tr}\theta + \delta + 2\lambda, \quad a^{-1}\mathrm{tr}\underline{\chi} = -\mathrm{tr}\theta + \delta + 2\lambda. \tag{1.8}$$

We define the null components of Riemannian curvature tensor relative to t-foliation,

$$\begin{aligned} \alpha_{AB} &= \mathbf{R}(L, e_A, L, e_B), &\qquad \beta_A &= \frac{1}{2}\mathbf{R}(e_A, L, \underline{L}, L), \\ \rho &= \frac{1}{4}\mathbf{R}(\underline{L}, L, \underline{L}, L), &\qquad \sigma &= \frac{1}{4}{}^\star\mathbf{R}(\underline{L}, L, \underline{L}, L), \\ \underline{\beta}_A &= \frac{1}{2}\mathbf{R}(e_A, \underline{L}, \underline{L}, L), &\qquad \underline{\alpha}_{AB} &= \mathbf{R}(\underline{L}, e_A, \underline{L}, e_B). \end{aligned} \tag{1.9}$$

We define also the mass aspect functions μ and $\underline{\mu}$ as follows

$$\mu = -\frac{1}{2}\mathbf{D}_3\mathrm{tr}\chi + \frac{a^2}{4}(\mathrm{tr}\chi)^2 - \varpi\mathrm{tr}\chi, \tag{1.10}$$

$$\underline{\mu} = \mathbf{D}_4\mathrm{tr}\underline{\chi} + \frac{1}{2}\mathrm{tr}\chi\cdot\mathrm{tr}\underline{\chi}. \tag{1.11}$$

Denote $\gamma_t := \gamma(t,\omega)$ the induced metric of $\mathbf{g}$ on S_t, relative to normal coordinates $\omega = (\omega_1, \omega_2)$ in the tangent space at p. Define the radius function of S_t to be $r(t) = \sqrt{(4\pi)^{-1}|S_t|}$ and define the metric $\overset{\circ}{\gamma}$ by $\overset{\circ}{\gamma} = r^{-2}\gamma$. We denote by $\gamma^{(0)}$ the canonical metric on $\mathbb{S}^2$. On each S_t we introduce the ratio of area elements

$$v_t(\omega) := \frac{\sqrt{|\gamma_t|}}{\sqrt{|\gamma^{(0)}|}}, \qquad \omega\in\mathbb{S}^2. \tag{1.12}$$

For smooth scalar functions f, the average of f on S_t is defined by $\bar{f} := \frac{1}{|S_t|}\int_{S_t} f d\mu_\gamma$. For any scalar functions f,

$$\int_{\mathcal{H}} f := \int_0^1\int_{S_t} fnad\mu_\gamma dt = \int_0^1\int_{|\omega|=1} nafv_t d\mu_{\mathbb{S}^2}dt.$$

We define L^p_ω norm for smooth functions f on S_t by $\|f\|^p_{L^p_\omega} = \int_{S_t}|f|^p d\mu_{\mathbb{S}^2}$, and define L^2 norm on null cone $\mathcal{H}$ for any smooth function f by

$$\|f\|^2_{L^2(\mathcal{H})} = \int_0^1\int_{S_t}|f|^2 nad\mu_\gamma dt.$$

For simplicity, we will suppress the $\mathcal{H}$ in the definition of norms on $\mathcal{H}$ whenever there occurs no confusion. Define for any S_t tangent tensor F the norm on $\mathcal{H}$

$$\mathcal{N}_1(F) = \|\nabla\!\!\!/_L F\|_{L^2} + \|\nabla\!\!\!/ F\|_{L^2} + \|r^{-1}F\|_{L^2}.$$

Define $\mathcal{R}(\mathcal{H})$, curvature flux on $\mathcal{H}$ relative to t-foliation, by

$$\mathcal{R}(\mathcal{H})^2 = \int_0^1\int_{S_t} an(|\alpha|^2 + |\beta|^2 + |\rho|^2 + |\sigma|^2 + |\underline{\beta}|^2)d\mu_\gamma dt.$$

For any S_t-tangent tensor field F we define the norm $\|F\|_{L^\infty_\omega L^2_t(\mathcal{H})}$ by

$$\|F\|^2_{L^\infty_\omega L^2_t} := \sup_{\omega\in\mathbb{S}^2}\int_0^1 |F|^2 nadt := \sup_{\omega\in\mathbb{S}^2}\int_{\Gamma_\omega}|F|^2 nadt.$$

The main result of this paper is the following

Theorem 1.1 (Main Theorem). *Let $(\mathbf{M}, \mathbf{g})$ be a smooth 3+1 Einstein vacuum spacetime foliated by Σ_t, the level hypersurfaces of a time function t with lapse function n. Consider an outgoing null hypersurface $\mathcal{H} = \cup_{0<t<1}S_t$ in $(\mathbf{M}, \mathbf{g})$ initiating from a point p, whose leaves are $S_t = \Sigma_t\cap\mathcal{H}$ and $t(p) = 0$. Assume $C^{-1} < n < C$ on $\mathcal{H}$ with certain positive constant C and assume that*

$$\mathcal{R}(\mathcal{H}) + \mathcal{N}_1(\not{\pi}) \le \mathcal{R}_0, \ \text{on } \mathcal{H} \tag{1.13}$$

with $\mathcal{R}_0$ sufficiently small. Then the following estimates hold true

$$\left\| tr\chi - \frac{2}{s}\right\|_{L^\infty(\mathcal{H})} \lesssim \mathcal{R}_0^2, \tag{1.14}$$

$$|a-1| \le \frac{1}{2}, \tag{1.15}$$

$$\left\| \int_0^1 |\hat{\chi}|^2 nadt \right\|_{L^\infty_\omega} + \left\| \int_0^1 |\zeta|^2 nadt \right\|_{L^\infty_\omega} \lesssim \mathcal{R}_0^2, \tag{1.16}$$

$$\left\| \int_0^1 |\underline{\zeta}|^2 nadt \right\|_{L^\infty_\omega} + \left\| \int_0^1 |\nu|^2 nadt \right\|_{L^\infty_\omega} \lesssim \mathcal{R}_0^2, \tag{1.17}$$

$$\|\nabla\!\!\!/ tr\chi\|_{\mathcal{P}^0} + \|\mu\|_{\mathcal{P}^0} + \|\nabla\!\!\!/ tr\chi\|_{L^2(\mathcal{H})} + \|\mu\|_{L^2(\mathcal{H})} \lesssim \mathcal{R}_0, \tag{1.18}$$

$$\mathcal{N}_1(\hat{\chi}) + \mathcal{N}_1(\zeta) + \mathcal{N}_1\left(tr\chi - \frac{2}{r}\right) + \mathcal{N}_1(tr\chi - (an)^{-1}\overline{antr\chi}) \lesssim \mathcal{R}_0, \tag{1.19}$$

$$\|tr\chi - \frac{2}{r}\|_{L^2_t L^\infty_\omega} + \|tr\chi - (an)^{-1}\overline{antr\chi}\|_{L^2_t L^\infty_\omega} \lesssim \mathcal{R}_0, \tag{1.20}$$

$$\left\| \sup_{t\le 1} |r^{3/2}\nabla\!\!\!/ tr\chi| \right\|_{L^2_\omega} + \left\| \sup_{t\le 1} r^{\frac{3}{2}}|\mu| \right\|_{L^2_\omega} + \left\| r^{1/2}\nabla\!\!\!/ tr\chi \right\|_{\mathcal{B}^0} + \|r^{1/2}\mu\|_{\mathcal{B}^0} \lesssim \mathcal{R}_0. \tag{1.21}$$

The Besov norms $\mathcal{P}^0$ and $\mathcal{B}^0$ appearing in the above statement will be defined by (4.8) and (4.9). Throughout this paper we will use the notation $A \lesssim B$ to mean $A \le C \cdot B$ for some appropriate constant C.

It is worthy to remark that (1.14) is the most important estimate in the main theorem and none of the estimates among (1.14)-(1.20) can be proved independent of others. Therefore we have to prove all of them simultaneously with a delicate bootstrap argument.

1.2. **Application.** By a standard rescaling argument, see [7, page 363], the estimates (1.14)-(1.21) in Theorem 1.1 can be rephrased as [12, Theorem 7, Proposition 14], which are the crucial components in the proof of an improved breakdown criterion for Einstein vacuum spacetime in CMC gauge, stated as follows

Theorem 1.2. [12, Theorem 1] *Let $(\mathcal{M}, \mathbf{g})$ be a globally hyperbolic development of Σ_{t_0} foliated by the CMC level hypersurfaces of a time function $t < 0$. Then the space-time together with the foliation Σ_t can be extended beyond any value $t_* < 0$ for which,*

$$\int_{t_0}^{t_*} \left(\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)} \right) dt = \mathcal{K}_0 < \infty. \tag{1.22}$$

Under the assumption (1.22) only, we have proved in [12, Section 3] that $C^{-1} < n < C$ with C depending only on t^*, $\mathcal{K}_0$ and the Bel-Robinson energy on the initial slice Σ_{t_0}. The condition (1.13) has also been achieved in [12, Theorems 5,6] under the assumption (1.22) with the help of a bootstrap argument (see [12, BA1–BA3]), in particular involved with energy estimate for the geometric wave equation of the second fundamental form k. Therefore we can apply Theorem 1.1 to close the proof for Theorem 1.2 ([12, Theorem 1]).

We recall that in [6, 7] the estimates in Theorem 1.1 were obtained under the assumption

$$\sup_{[t_0, t_*)} \left(\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)} \right) = \Lambda_0 < \infty \tag{1.23}$$

combined with the same assumptions on $\mathcal{R}(\mathcal{H})$ and n, both of which can be obtained under (1.23) or the weaker assumption (1.22). The key improvement in Theorem 1.1 lies in that it relies on the weaker assumption

$$\mathcal{N}_1(\pi\!\!\!/) \le \mathcal{R}_0, \tag{1.24}$$

which comes naturally as a consequence of (1.22).

Due to the much weaker assumption (1.24), we no longer can adopt the approach contained in [5, 6, 7] to control Ricci coefficients and $|a-1|$. For instance, let us consider the estimate on $|a-1|$. As required in [7] and [12], we need to show that

$$|a-1| \le \frac{1}{2}. \tag{1.25}$$

For $q > 1$, an assumption of

$$\left(\int_{t_0}^{t_*} (\|k\|^q_{L^\infty(\Sigma)} + \|\nabla \log n\|^q_{L^\infty(\Sigma)})\right)^{\frac{1}{q}} < \Lambda_0 < \infty \tag{1.26}$$

by rescaling takes the following form on null cone $\mathcal{H}$

$$\|k\|_{L_t^q L_x^\infty(\mathcal{H})} + \|\nabla \log n\|_{L_t^q L_x^\infty(\mathcal{H})} < \mathcal{R}_0,$$

which by (1.5) and (1.6) immediately gives

$$\|\nu\|_{L_t^q L_x^\infty(\mathcal{H})} + \|\underline{\zeta}\|_{L_t^q L_x^\infty(\mathcal{H})} \lesssim \mathcal{R}_0. \tag{1.27}$$

In view of the definition $\nu := -\nabla\!\!\!/_L a$ and $a(p) = 1$, we can obtain (1.25) by integrating along any null geodesic Γ_ω as long as $\mathcal{R}_0$ is sufficiently small.

If $q = 1$, the above simple argument fails due to the fact that by recaling, there holds

$$\|k\|_{L_t^1 L_x^\infty(\mathcal{H})} + \|\nabla \log n\|_{L_t^1 L_x^\infty(H)} < \mathcal{K}_0$$

which fails to be small. Thus, it is impossible to derive (1.27) immediately from assumption. Theorem 1.1 however provides the trace estimate for $\|\nu\|_{L_x^\infty L_t^2(\mathcal{H})}$ in (1.17) which is strong enough to guarantee $|a-1| \le \frac{1}{2}$.

As remarked after the statement of Theorem 1.1, estimates such as

$$\|\underline{\zeta}\|_{L_\omega^\infty L_t^2(\mathcal{H})} \lesssim \mathcal{R}_0 \cdots$$

are indispensable for establishing (1.14), (1.16) and estimates on μ and $\nabla\!\!\!/ \mathrm{tr}\chi$, all of which were employed to prove breakdown criterion in [7] and [12]. The above estimate for $\underline{\zeta}$ can not directly follow from the assumption (1.26) with $q < 2$.

It is well known that the embedding $H^1(S_t) \hookrightarrow L^\infty(S_t)$ fails. Therefore the assumption (1.24), which is a consequence of (1.22) by energy estimate, can neither control $\|\underline{\zeta}, \nu\|_{L_t^2 L_x^\infty}$ immediately nor by Sobolev embedding. This forces us to estimate the weaker norm $\|\cdot\|_{L_x^\infty L_t^2(\mathcal{H})}$, which, according to our experience, would succeed only when special structures for $\nabla\!\!\!/ \underline{\zeta}$ and $\nabla\!\!\!/ \nu$ can be found.

1.3. **Comparison to geodesic foliation.** Let us draw comparisons between geodesic foliation and time foliation on null hypersurfaces as follows.

(1) In the case of geodesic foliation ([2, 11]), the complete set of estimates in the main theorem can be obtained under the small curvature flux only. However in time foliation, (1.13) contains one more assumption (1.24). In order to understand the reason for assuming (1.24), let us sketch the approach to derive (1.17). We will prove and employ the sharp trace inequality (see Theorem 5.1) on null cones with time foliation, which lied in the heart of [2, 4, 3] for null hypersurfaces with geodesic foliation. To implement this idea, it is a must to control $\mathcal{N}_1(\nu, \underline{\zeta})$. In view of (1.5) and (1.6), ν and $\underline{\zeta}$ are combinations of elements of $\pi\!\!\!/$, therefore the assumption (1.24) guarantees the necessary control on $\nu, \underline{\zeta}$.

(2) The identity $\zeta + \underline{\zeta} = 0$ only holds on null hypersurface $\mathcal{H}$ with geodesic foliation. Therefore, in the case of time foliation, the estimates for $\underline{\zeta}$ are no longer identical to those for ζ. Note that $\underline{\zeta}$ and ζ are on an equal footing in structure equations (2.4) and (2.15), we need $\mathcal{N}_1(\cdot)$ and $\|\cdot\|_{L_x^\infty L_t^2(\mathcal{H})}$ estimates for both of them. ζ will be treated in the same fashion as in geodesic foliation, while the estimate for $\|\underline{\zeta}\|_{L_x^\infty L_t^2(\mathcal{H})}$ requires further study on structure equations. Similar to (2.8) and (2.9), the quantity $\underline{\mu}$ is connected to $\underline{\zeta}$ by the Hodge system, (2.5) and (2.7),

$$\begin{cases} \text{div}\zeta = -\check{\rho} - \mu + \cdots \\ \text{curl}\,\zeta = \check{\sigma}, \end{cases} \qquad \begin{cases} \text{div}\underline{\zeta} = -\check{\rho} + \frac{1}{2}\underline{\mu} + \cdots \\ \text{curl}\,\underline{\zeta} = -\check{\sigma}. \end{cases}$$

However $\underline{\mu}$ fails to satisfy a similar transport equation as (2.15) for μ, $\underline{\mu}$ consequently does not verify $\|\cdot\|_{\mathcal{P}^0}$ estimate as μ, the treatment of $\underline{\zeta}$ is therefore different from ζ. Note that if there holds the following decomposition for $\nabla\!\!\!/\underline{\zeta}$,

$$\nabla\!\!\!/\underline{\zeta} = \nabla\!\!\!/_L P + E \tag{1.28}$$

with P and E satisfying appropriate estimates, we may rely on the sharp trace inequality to estimate $\|\underline{\zeta}\|_{L_\omega^\infty L_t^2}$. To obtain the important structure (1.28), we first derive a refined Hodge system, (2.16) and (2.7),

$$\begin{cases} \text{div}\underline{\zeta} = -\check{\rho} + L(a\delta + 2a\lambda) + \cdots \\ \text{curl}\,\underline{\zeta} = -\check{\sigma}. \end{cases} \tag{1.29}$$

The pair of quantities $(\check{\rho}, \check{\sigma})$ can be decomposed in the same way as contained in [2, 11]. We set $\mathcal{D}_1 : F \to (\text{div}F, \text{curl}\,F)$ for any smooth S_t tangent 1-form F. (1.28) then will be obtained from (1.29) by commuting $\nabla\!\!\!/_L$ with $\nabla\!\!\!/\mathcal{D}_1^{-1}$. The new commutator $[\nabla\!\!\!/_L, \nabla\!\!\!/\mathcal{D}_1^{-1}](a\delta + 2a\lambda)$ will be decomposed in Proposition 6.1 together with other commutators arising in control of $\|\hat{\chi}, \zeta\|_{L_x^\infty L_t^2(\mathcal{H})}$.

Recall that in order to control $|a-1| < 1/2$ in (1.25), we need to estimate $\|\nu\|_{L_\omega^\infty L_t^2}$, which is a quantity that does not arise in the case of geodesic foliation. We again rely on sharp trace inequality to derive $\|\nu\|_{L_\omega^\infty L_t^2}$, which requires another remarkable structure of the form,

$$\nabla\!\!\!/\nu = \nabla\!\!\!/_L P + E.$$

This will be done by deriving the transport equation for $\nabla\!\!\!/ a$, i.e. (2.17),

$$\nabla\!\!\!/\nu = -\nabla_L(\nabla\!\!\!/ a) - \frac{1}{2}\text{tr}\chi\nabla\!\!\!/ a + \cdots.$$

To prove sharp trace inequality and to control P and E actually dominate the article. Further comparison on technical details will be made in Section 2.

1.4. **Organization of the paper.** This paper is organized as follows. In Section 2, we start with providing all the structure equations and making bootstrap assumptions. Then by using structure equations (2.1)-(2.18) and Sobolev embedding, we establish a series of preliminary estimates including weakly spherical property for the metric $\overset{\circ}{\gamma}$. In Section 3, we prove $\|\Lambda^{-\alpha}\underline{K}\|_{L_t^\infty L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0$ with $\alpha \ge 1/2$ and establish a series of elliptic estimates. In Section 4, we briefly review the theory of geometric Littlewood Paley decomposition (GLP) and define Besov norms. We give the equivalence relation on Besov norms in Proposition 4.2 and the reduction argument in Lemma 4.1 based on the weakly spherical property for $\overset{\circ}{\gamma}$. With the help of these two arguments, in Section 5, we prove the sharp trace theorem, i.e.

Theorem 5.1. In order to obtain the structure required in Theorem 5.1, the commutators involved are decomposed in Proposition 6.1, which is the main purpose of Section 6. In Section 7, we estimate $\|\hat{\chi}, \underline{\zeta}, \zeta, \nu\|_{L_\omega^\infty L_t^2}$ by using Theorem 5.1 and Proposition 6.1. In Section 8, we prove dyadic Sobolev inequalities and (6.76) in Theorem 6.1.

2. **Preliminary estimates**

2.1. **Structure equations.** The proof of Main Theorem relies crucially on the following set of structure equations. We will prove (2.16) and (2.17) and refer the reader to [1, Chapter 11] and [2, Section 2] for the derivation of all other formulae.

$$\frac{d\mathrm{tr}\chi}{ds} + \frac{1}{2}(\mathrm{tr}\chi)^2 = -|\hat{\chi}|^2, \tag{2.1}$$

$$\frac{d\hat{\chi}_{AB}}{ds} + \mathrm{tr}\chi\hat{\chi}_{AB} = -\alpha_{AB}, \tag{2.2}$$

$$\frac{d}{ds}\zeta_A = -\chi_{AB}\zeta_B + \chi_{AB}\underline{\zeta}_B - \beta_A, \tag{2.3}$$

$$\frac{d}{ds}\nabla\!\!\!/\mathrm{tr}\chi + \frac{3}{2}\mathrm{tr}\chi\nabla\!\!\!/\mathrm{tr}\chi = -\hat{\chi}\cdot\nabla\!\!\!/\mathrm{tr}\chi - 2\hat{\chi}\cdot\nabla\!\!\!/\hat{\chi} - (\zeta+\underline{\zeta})(|\hat{\chi}|^2 + \frac{1}{2}(\mathrm{tr}\chi)^2), \tag{2.4}$$

$$\frac{d}{ds}\mathrm{tr}\underline{\chi} + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} = 2\mathrm{div}\underline{\zeta} - \hat{\chi}\cdot\underline{\hat{\chi}} + 2|\underline{\zeta}|^2 + 2\rho, \tag{2.5}$$

$$\mathrm{div}\hat{\chi} = \frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\chi\cdot\zeta - \hat{\chi}\cdot\zeta - \beta, \tag{2.6}$$

$$\mathrm{curl}\,\underline{\zeta} = \frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}} - \sigma, \tag{2.7}$$

$$\mathrm{div}\zeta = -\mu - \check{\rho} - |\zeta|^2 + \frac{1}{2}a\delta\mathrm{tr}\chi + a\lambda\mathrm{tr}\chi \tag{2.8}$$

$$\mathrm{curl}\,\zeta = \check{\sigma} \tag{2.9}$$

In what follows, we record null Bianchi equations

$$\nabla\!\!\!/_L\beta_A = \mathrm{div}\alpha - 2\mathrm{tr}\chi\beta_A + (2\zeta_A + \underline{\zeta}_A)\alpha_{AB} \tag{2.10}$$

$$L\check{\rho} + \frac{3}{2}\mathrm{tr}\chi\cdot\check{\rho} = \mathrm{div}\beta + (\zeta + 2\underline{\zeta})\cdot\beta - \frac{1}{2}\hat{\chi}\cdot(\nabla\!\!\!/\widehat{\otimes}\underline{\zeta} - \frac{1}{2}\mathrm{tr}\underline{\chi}\cdot\hat{\chi} + \underline{\zeta}\widehat{\otimes}\underline{\zeta}), \tag{2.11}$$

$$L\check{\sigma} + \frac{3}{2}\mathrm{tr}\chi\cdot\check{\sigma} = -\,\mathrm{curl}\,\beta - (\zeta + 2\underline{\zeta})\wedge\beta - \frac{1}{2}\hat{\chi}\wedge(\nabla\!\!\!/\widehat{\otimes}\underline{\zeta} + \underline{\zeta}\widehat{\otimes}\underline{\zeta}), \tag{2.12}$$

$$\nabla\!\!\!/_L\underline{\beta} + \mathrm{tr}\chi\underline{\beta} = -\nabla\!\!\!/\rho + (\nabla\!\!\!/\sigma)^\star + 2\underline{\hat{\chi}}\cdot\beta - 3(\underline{\zeta}\rho - \underline{\zeta}^\star\sigma) \tag{2.13}$$

where

$$\check{\rho} = \rho - \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}}, \quad \check{\sigma} = \sigma - \frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}}. \tag{2.14}$$

Moreover, there hold

$$\begin{aligned} L(\mu)+\mathrm{tr}\chi\mu &= 2\hat{\chi}\cdot\nabla\!\!\!\!/\,\zeta+(\zeta-\underline{\zeta})\cdot(\nabla\!\!\!\!/_A\mathrm{tr}\chi+\mathrm{tr}\chi\zeta_A)-\frac{1}{2}\mathrm{tr}\chi(\hat{\chi}\cdot\underline{\hat{\chi}}-2\rho+2\underline{\zeta}\cdot\zeta)\\ &+2\zeta\cdot\hat{\chi}\cdot\zeta+\left(\frac{1}{4}a^2\mathrm{tr}\chi-a(\delta+2\lambda)\right)|\hat{\chi}|^2-\frac{1}{2}a\nu(\mathrm{tr}\chi)^2, \end{aligned} \tag{2.15}$$

$$L(a\delta+2a\lambda)+\frac{3}{2}a\delta\mathrm{tr}\chi=\mathrm{div}\underline{\zeta}+\check{\rho}+|\underline{\zeta}|^2+a\mathrm{tr}\chi\nabla_N\log n, \tag{2.16}$$

$$\nabla\!\!\!\!/_L(\nabla\!\!\!\!/\,a)+\frac{1}{2}\mathrm{tr}\chi\nabla\!\!\!\!/\,a=-\nabla\!\!\!\!/\,\nu-\hat{\chi}\cdot\nabla\!\!\!\!/\,a-(\underline{\zeta}+\zeta)\cdot\nu. \tag{2.17}$$

The Gauss curvature K on each S_t verifies

$$K=-\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}+\frac{1}{2}\underline{\hat{\chi}}\cdot\hat{\chi}-\rho. \tag{2.18}$$

The following two commutation formulas hold:

(i) For any scalar functions U,

$$\frac{d}{ds}\nabla\!\!\!\!/_A U+\chi_{AB}\nabla\!\!\!\!/_B U=\nabla\!\!\!\!/_A F+(\underline{\zeta}+\zeta)F, \tag{2.19}$$

where $F=\frac{d}{ds}U$.

(ii) For S_t tangent 1-form U_A satisfying $\frac{dU_A}{ds}=F_A$, there holds

$$\frac{d}{ds}\mathrm{div}U+\chi_{AB}\nabla\!\!\!\!/_A U_B=\mathrm{div}F+(\zeta+\underline{\zeta})\cdot F+(\frac{1}{2}\mathrm{tr}\chi\underline{\zeta}_A-\hat{\chi}_{AB}\underline{\zeta}_B+\beta_A)U_A. \tag{2.20}$$

Proof of (2.16). In view of (2.5) and (2.1),

$$\begin{aligned} L(a^{-1}\mathrm{tr}\underline{\chi}) &= L(\mathrm{tr}\underline{\chi})a^{-1}+\mathrm{tr}\underline{\chi}L(a^{-1})\\ &= a^{-1}(2\mathrm{div}\underline{\zeta}+2\check{\rho}+2|\underline{\zeta}|^2-\frac{1}{2}\mathrm{tr}\chi\cdot\mathrm{tr}\underline{\chi})-a^{-2}L(a)\mathrm{tr}\underline{\chi},\\ L(a\mathrm{tr}\chi) &= L(\mathrm{tr}\chi)a+\mathrm{tr}\chi L(a)=-a(|\hat{\chi}|^2+\frac{1}{2}(\mathrm{tr}\chi)^2)+L(a)\mathrm{tr}\chi \end{aligned}$$

By using (1.8),

$$\begin{aligned} L(2\delta+4\lambda) &= a^{-1}(2\mathrm{div}\underline{\zeta}+2\check{\rho}+2|\underline{\zeta}|^2)-\frac{1}{2}\mathrm{tr}\chi(a^{-1}\mathrm{tr}\underline{\chi}+a\mathrm{tr}\chi)+L\log a(a\mathrm{tr}\chi-a^{-1}\mathrm{tr}\underline{\chi})\\ &= a^{-1}2(\mathrm{div}\underline{\zeta}+\check{\rho}+|\underline{\zeta}|^2)-\mathrm{tr}\chi(\delta+2\lambda)+2L(\log a)\mathrm{tr}\theta, \end{aligned}$$

we can obtain with the help of (1.8) and (1.5)

$$\begin{aligned} L(a\delta) &= aL(\delta)+\delta L(a)\\ &= (\mathrm{div}\underline{\zeta}+\check{\rho}+|\underline{\zeta}|^2)-\frac{1}{2}a\mathrm{tr}\chi(\delta+2\lambda)+L(a)(\mathrm{tr}\theta+\delta)-2aL(\lambda)\\ &= \mathrm{div}\underline{\zeta}+\check{\rho}+|\underline{\zeta}|^2+a\mathrm{tr}\chi(-\frac{3}{2}\delta+\nabla_N\log n)-L(2a\lambda) \end{aligned}$$

which gives the desired formula. □

Proof of (2.17). Recall that by $L(a)=-\nu$, combined with (2.19)

$$[\nabla\!\!\!\!/_L,\nabla\!\!\!\!/\,]a=-\chi\cdot\nabla\!\!\!\!/\,a-(\zeta+\underline{\zeta})\nu$$

we may obtain relative to orthornomal frame on S_t,

$$\nabla\!\!\!\!/_L\nabla\!\!\!\!/_B a+\frac{1}{2}\mathrm{tr}\chi\nabla\!\!\!\!/_B a=-\nabla\!\!\!\!/_B\nu-\hat{\chi}_{BC}\cdot\nabla\!\!\!\!/_C a-(\zeta_B+\underline{\zeta}_B)\nu$$

This completes the proof. $\square$

Let $\mathcal{D}_t$ denote $\frac{d}{dt}$ along a null geodesics initiating from p. In view of (1.2), $\mathcal{D}_t = an\frac{d}{ds}$. In comparison with geodesic foliation, the term $(\zeta+\underline{\zeta})\nabla\!\!\!/_L U$ in (2.19) is no longer trivial due to $\zeta+\underline{\zeta}\neq 0$. This term however can be avoided if we consider the commutator $[\mathcal{D}_t,\nabla\!\!\!/]U$ instead. Similarly, in view of [1, Lemma 13.1.2], it is simpler to consider $[\mathcal{D}_t,\nabla\!\!\!/]F$ than $[\nabla\!\!\!/_L,\nabla\!\!\!/]F$ for S-tangent tensor fields F.

Proposition 2.1. *For any smooth scalar function f,*

$$[\mathcal{D}_t,\nabla\!\!\!/]f = -an\chi\cdot\nabla\!\!\!/ f \tag{2.21}$$

In view of (2.21), (2.20) and (2.6), there holds

$$\begin{aligned}[\mathcal{D}_t,\Delta\!\!\!/]f &= -an tr\chi\Delta\!\!\!/ f - 2an\hat{\chi}\cdot\nabla\!\!\!/^2 f + 2an\beta\cdot\nabla\!\!\!/ f - 2an\underline{\zeta}\hat{\chi}\cdot\nabla\!\!\!/ f\\ &\quad - an\zeta tr\chi\nabla\!\!\!/ f - an\nabla\!\!\!/ tr\chi\nabla\!\!\!/ f.\end{aligned}$$

Combining [11, P.288] with the comparison formulas in [2, Section 2], we have

Lemma 2.1.

$$V,\nabla\!\!\!/ a, r\nabla\!\!\!/ tr\chi, r^2\mu\to 0 \ \textit{as}\ t\to 0,\quad \lim_{t\to 0}\|\hat{\chi},\zeta,\underline{\zeta},\nu\|_{L^\infty(S_t)}<\infty$$

For S tangent tensor fields F on $\mathcal{H}$, we introduce the following norms. For $1\le p,q\le\infty$ we define the $L_t^qL_x^p$ norm on $\mathcal{H}$

$$\|F\|_{L_t^qL_x^p} := \left(\int_0^1\left(\int_{|\omega|=1}|F(t,\omega)|^p nav_t d\mu_{\mathbb{S}^2}\right)^{\frac{q}{p}}dt\right)^{\frac{1}{q}}$$

and the $L_x^pL_t^\infty$ norm

$$\|F\|^p_{L_x^pL_t^\infty} := \int_{\mathbb{S}^2}\sup_{t\in\Gamma_\omega}\left(v_t|F|^p\right)d\mu_{\mathbb{S}^2}.$$

2.2. **Notations and Bootstrap assumptions.** We fix the following conventions

- $\pi\!\!\!/$ denotes the collection of $\hat{\eta}$, ϵ, δ, $\nabla_N\log n$, $\nabla\!\!\!/\log n$, λ,
- $\iota := \mathrm{tr}\chi-\frac{2}{r}, V := \mathrm{tr}\chi-\frac{2}{s},\quad \kappa := \mathrm{tr}\chi-(an)^{-1}\overline{an\mathrm{tr}\chi}$,
- A denotes the collection of $\hat{\chi}$, ζ, $\underline{\zeta}$, ν,
- $\underline{A}$ denotes the collection of A and $\underline{\hat{\chi}}$, $\nabla\!\!\!/\log a$, $\pi\!\!\!/$,
- The pair of quantities $(M,\mathcal{D}_0M)$ denotes either $(\nabla\!\!\!/\mathrm{tr}\chi,\nabla\!\!\!/\hat{\chi})$, or $(\mu,\nabla\!\!\!/\zeta)$,
- R_0 denotes the collection of α, $\beta,\rho,\sigma,\underline{\beta}$,
- $\bar{R}$ denotes the collection of $R_0,\mathrm{tr}\chi\underline{A}$, $A\cdot\underline{A}$,
- $\tilde{R}$ denotes the collection of $\bar{R}$ and $\nabla\!\!\!/ A$,
- $\mathcal{H}_t := \cup_{t'\in[0,t]}S_{t'}$, with $\quad 0<t\le 1$,
- $S := S_t,\ \overset{\circ}{\gamma} := r^{-2}\gamma,\ \gamma^{(0)} := \gamma_{\mathbb{S}^2},\ \underline{K} := K-\frac{1}{r^2}$.

Assumption 2.1. *We make the following bootstrap assumption:*

$$\|V\|_{L^\infty(\mathcal{H})}\le\Delta_0,\quad \|\hat{\chi},\nu,\zeta,\underline{\zeta}\|_{L_\omega^\infty L_t^2(\mathcal{H})}\le\Delta_0,\quad |a-1|\le\frac{1}{2} \tag{BA1}$$

where we can assume that $0<\mathcal{R}_0<\Delta_0<1/2$.

The goal is to improve the inequalities in BA1 with the Δ_0 replaced by $\Delta_0^2+\mathcal{R}_0$, and $|a-1|\le \frac14$. When $\mathcal{R}_0$ is sufficiently small, $\Delta_0^2+\mathcal{R}_0<\frac12\Delta_0$ can be achieved. We will start with deriving estimates for $\mathcal{N}_1(\underline{A})$ by establishing related estimates for $M=\nabla\!\!\!/ \mathrm{tr}\chi,\mu$, which will be contained in Propositions 2.4 and 2.5. Then we prove that κ and ι verify stronger estimates than A, which can be seen in (2.84) and Proposition 2.7. At last we prove $(S_t,\overset{\circ}{\gamma})$ is weakly spherical.

2.3. **Estimates for $\mathcal{N}_1(\underline{A})$, $\|r^{1/2}M\|_{L_x^2L_t^\infty}$ and $\|M\|_{L^2}$.** Recall a few results that have been proved in [12] and [2, 3, 10].

Proposition 2.2. [12] *Under the assumption BA1, there hold*

$$C^{-1}\le v_t/s^2\le C,\quad C^{-1}<\frac{r}{s}<C, \tag{2.22}$$

where C is a positive constant.

It is easy to derive from $|V|\le\Delta_0$ in BA1 and Proposition 2.2 that

$$|s\mathrm{tr}\chi|+|r\mathrm{tr}\chi|\le C \tag{2.23}$$

and from $|a-1|\le\frac12$ in BA1 and $C^{-1}<n<C$ that

$$C^{-1}<an<C, \tag{2.24}$$

with C positive constants.

With the help of Proposition 2.2, there hold the following simple inequalities by Sobolev embedding in 2-D slices $S=S_t$.

- Let $\mathcal{O}sc(f):=f-\bar f$ for any smooth function f on S where $\bar f=\frac{1}{|S|}\int_S f d\mu_\gamma$, there holds the Poincare inequality

$$\|r^{-1}\mathcal{O}sc(f)\|_{L^2(S)}\lesssim\|\nabla\!\!\!/ f\|_{L^2(S)}. \tag{Poin}$$

- For a smooth function Ω on S with vanishing mean, there holds the following Sobolev inequality (see [3])

$$\|\Omega\|_{L^\infty(S)}\lesssim\|\nabla\!\!\!/^2\Omega\|_{L^1(S)}+\|\nabla\!\!\!/\Omega\|_{L^2(S)}, \tag{GaNi}$$

  which implies

$$r^{-1}\|\Omega\|_{L^\infty(S)}\lesssim\|\nabla\!\!\!/^2\Omega\|_{L^2(S)}+r^{-1}\|\nabla\!\!\!/\Omega\|_{L^2(S)}, \tag{2.25}$$

$$\|r^{-1}\Omega\|_{L_t^2L_x^\infty}\lesssim\|\nabla\!\!\!/^2\Omega\|_{L^2}+\|r^{-1}\nabla\!\!\!/\Omega\|_{L^2}. \tag{2.26}$$

- Let F be a S tangent tensor field, (see [3])

$$\|F\|_{L_x^p(S)}\lesssim\|\nabla\!\!\!/ F\|_{L_x^2(S)}^{1-\frac2p}\|F\|_{L_x^2(S)}^{\frac2p}+\|r^{-1+\frac2p}F\|_{L_x^2(S)},\ \text{with } 2<p<\infty. \tag{Sob}$$

- Let F be a S tangent tensor field, there hold (see [2, 11])

$$\|r^{-1/2}F\|_{L_x^2L_t^\infty}+\|F\|_{L_x^4L_t^\infty}+\|F\|_{L^6}\lesssim\mathcal{N}_1(F), \tag{SobM1}$$

$$\|r^{-\frac12}F\|_{L^\infty}+\|r^{-1}F\|_{L_t^2L_x^\infty}\lesssim\mathcal{N}_2(F), \tag{SobM2}$$

  where

$$\mathcal{N}_2(F):=\|r^{-2}F\|_{L^2}+\|r^{-1}\nabla\!\!\!/_L F\|_{L^2}+\|r^{-1}\nabla\!\!\!/ F\|_{L^2}+\|\nabla\!\!\!/\mathcal{D}_tF\|_{L^2}+\|\nabla\!\!\!/^2F\|_{L^2}. \tag{2.27}$$

  By interpolation,

$$\|r^{-\frac1b}F\|_{L_t^bL_x^4}+\|r^{-\frac1q-\frac12}F\|_{L_t^qL_x^2}\lesssim\mathcal{N}_1(F),\ \text{with } b\ge4,\ q\ge2. \tag{2.28}$$

Lemma 2.2.

$$\|V\|_{L^\infty} \lesssim \Delta_0^2 \tag{2.29}$$

Proof. This can be obtained by integrating along Γ_ω the equation (2.1), i.e.

$$\frac{d}{ds}V + \frac{2}{s}V = -\frac{1}{2}V^2 - |\hat{\chi}|^2$$

with the help of Proposition 2.2, Lemma 2.1 and $\|V\|_{L^\infty} + \|\hat{\chi}\|_{L_\omega^\infty L_t^2} \le \Delta_0$ in BA1. $\Box$

Lemma 2.3. *For a S tangent tensor field F verifying*

$$\nabla\!\!\!/_L F + \frac{p}{2} tr\chi F = G \cdot F + H \tag{2.30}$$

with $p \ge 1$ certain integer, if $\lim_{t\to 0} r(t)^p F = 0$ and $\|G\|_{L_\omega^\infty L_t^2} \lesssim \Delta_0$, then the following estimate holds

$$|F| \lesssim v_t^{-\frac{p}{2}} \int_0^t v_{t'}{}^{\frac{p}{2}} |H| n a dt'. \tag{2.31}$$

We will constantly use the Hardy-Littlewood inequality for scalar f on $\mathcal{H}$,

$$\left\| \frac{1}{s}\int_0^s |f| \right\|_{L_s^2} \lesssim \|f\|_{L_s^2} \tag{2.32}$$

With the help of Lemmas 2.1 and 2.3, (2.2), (2.3), BA1, (2.32) and (1.13) we obtain

Lemma 2.4.

$$\|r^{-1}\hat{\chi}, r^{-1}\zeta\|_{L^2} + \|r^{-1/2}\hat{\chi}, r^{-1/2}\zeta\|_{L_x^2 L_t^\infty} \lesssim \mathcal{R}_0, \tag{2.33}$$

$$\|\nabla\!\!\!/_L \hat{\chi}, \nabla\!\!\!/_L \zeta\|_{L^2} \lesssim \mathcal{R}_0 + \Delta_0^2. \tag{2.34}$$

In view of Lemma 2.4, (1.13) and (**SobM1**), by definition of elements of $\underline{A}$, we can summarize the estimates for $\underline{A}$,

Proposition 2.3.

$$\|r^{-1}\underline{A}\|_{L^2} + \|r^{-1/2}\underline{A}\|_{L_x^2 L_t^\infty} + \|\nabla\!\!\!/_L \underline{A}\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

For the proofs of Lemmas 2.3 and 2.4 and Proposition 2.3, see [12].

With the help of Proposition 2.3 and Lemma 2.3, we prove the following result under the assumption of BA1.

Lemma 2.5.

$$\|\nabla\!\!\!/ \log s\|_{L_\omega^\infty L_t^2} \lesssim \Delta_0, \tag{2.35}$$

$$\|\nabla\!\!\!/ \log s\|_{L_t^2 L_\omega^2} + \|s^{1/2}\nabla\!\!\!/ \log s\|_{L_\omega^2 L_t^\infty} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{2.36}$$

$$\|\mathcal{O}sc(\frac{1}{s})\|_{L_t^2 L_\omega^2} + \|s^{\frac{1}{2}}\mathcal{O}sc(\frac{1}{s})\|_{L_t^\infty L_\omega^2} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{2.37}$$

$$\|\kappa\|_{L_t^2 L_\omega^2} + \|r^{1/2}\kappa\|_{L_t^\infty L_\omega^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.38}$$

Proof. Apply (2.19) to $U = s$, we can derive the transport equation

$$\frac{d}{ds}\nabla\!\!\!\!/_A(s) + \frac{1}{2}\mathrm{tr}\chi\nabla\!\!\!\!/_A(s) = -\hat{\chi}_{AB}\nabla\!\!\!\!/_B(s) + \zeta_A + \underline{\zeta}_A. \tag{2.39}$$

Note that $e_A(s) \to 0,$ as $t \to 0$,[2] in view of Lemma 2.3 with $G = \hat{\chi}$ and BA1, we can derive by integrating along a null geodesic Γ_ω initiating from vertex,

$$|s^{-1}\nabla\!\!\!\!/(s)(t)| \lesssim \frac{1}{s}v_t^{-\frac{1}{2}}\int_0^t v_{t'}^{\frac{1}{2}}|\zeta + \underline{\zeta}|nadt'. \tag{2.40}$$

Taking L_t^2 norm first then $L_\omega^\infty(\mathbb{S}^2)$, with the help of BA1,

$$\|s^{-1}\nabla\!\!\!\!/(s)\|_{L_\omega^\infty L_t^2} \lesssim \|\zeta + \underline{\zeta}\|_{L_\omega^\infty L_t^2} \lesssim \Delta_0$$

which gives (2.35).

By taking L_t^2 norm first then $L_\omega^2(\mathbb{S}^2)$, we can obtain from (2.40) by using (2.32) that

$$\|s^{-1}\nabla\!\!\!\!/(s)\|_{L_\omega^2 L_t^2} \lesssim \|r^{-1}(\zeta + \underline{\zeta})\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0,$$

where for the last inequality we employed $\|r^{-1}\underline{A}\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0$ in Proposition 2.3. Similarly

$$\|s^{-1/2}\nabla\!\!\!\!/ s\|_{L_\omega^2 L_t^\infty} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Hence (2.36) is proved.

Applying (**Poin**) to $f = \frac{1}{s}$, (2.37) follows as a consequence of (2.36).

According to definition, we can derive

$$\begin{aligned}\kappa &= \mathrm{tr}\chi - \frac{2}{s} - (an)^{-1}\overline{an(\mathrm{tr}\chi - \frac{2}{s})} + \frac{2}{s}(1 - (an)^{-1}\overline{an}) \\ &\quad + 2\mathcal{O}sc(\frac{1}{s})(an)^{-1}\overline{an} - 2(an)^{-1}\overline{s^{-1}\mathcal{O}sc(an)}.\end{aligned} \tag{2.41}$$

By (**Poin**), (2.24) and also in view of Propositions 2.2 and 2.3, we obtain

$$\|s^{-1}\mathcal{O}sc(an)\|_{L_t^2 L_\omega^2} \lesssim \|\nabla\!\!\!\!/ \log(an)\|_{L_t^2 L_\omega^2} \lesssim \|r^{-1}(\zeta + \underline{\zeta})\|_{L^2(\mathcal{H})} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{2.42}$$

and similarly

$$\|s^{-\frac{1}{2}}\mathcal{O}sc(an)\|_{L_t^\infty L_\omega^2} \lesssim \|r^{\frac{1}{2}}(\zeta + \underline{\zeta})\|_{L_t^\infty L_\omega^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.43}$$

Using (2.24) and (2.42), the last term in (2.41) can be estimated as follows

$$\|(an)^{-1}\overline{s^{-1}\mathcal{O}sc(an)}\|_{L_t^2} \lesssim \|s^{-1}\mathcal{O}sc(an)\|_{L_t^2 L_\omega^1} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.44}$$

Combining (2.42), (2.44) and (2.29), we obtain

$$\|r^{-1}\kappa\|_{L^2} \lesssim \|r^{-1}\mathcal{O}sc(\frac{1}{s})\|_{L^2} + \Delta_0^2 + \mathcal{R}_0 \lesssim \Delta_0^2 + \mathcal{R}_0.$$

where, for the last inequality, we employed (2.37).

By (2.37), (2.43) and (2.29), we can get

$$\begin{aligned}\|r^{\frac{1}{2}}\kappa\|_{L_t^\infty L_\omega^2} &\le \|r^{\frac{1}{2}}\mathcal{O}sc(\frac{1}{s})\|_{L_t^\infty L_\omega^2} + \|(an)^{-1}r^{-\frac{1}{2}}\mathcal{O}sc(an)\|_{L_t^\infty L_\omega^1} + |V| \\ &\quad + \|s^{-\frac{1}{2}}((\overline{an})^{-1}an - 1)\|_{L_t^\infty L_\omega^2} \lesssim \Delta_0^2 + \mathcal{R}_0.\end{aligned}$$

The proof is complete. $\square$

[2] This initial condition can be easily checked by using the comparison formulas in [2, Section 2].

Lemma 2.6.

$$\left\| \overline{tr\chi} - \frac{2}{r} \right\|_{L_t^2} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{2.45}$$

$$\left\| tr\chi - \frac{2}{r} \right\|_{L_t^2 L_\omega^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.46}$$

Proof. We can derive the transport equation

$$\frac{d}{ds}(r(\overline{tr\chi} - \frac{2}{r})) = (an)^{-1}\frac{r}{2}\overline{an\mathrm{tr}\chi\kappa} - r(an)^{-1}\overline{an|\hat{\chi}|^2}. \tag{2.47}$$

by combining

$$\frac{d}{ds}r = (an)^{-1}\frac{r}{2}\overline{an\mathrm{tr}\chi}. \tag{2.48}$$

with

$$\frac{d}{ds}\overline{\mathrm{tr}\chi} = -(an)^{-1}\overline{an\mathrm{tr}\chi}\cdot\overline{\mathrm{tr}\chi} + (an)^{-1}\overline{an(\frac{1}{2}(\mathrm{tr}\chi)^2 - |\hat{\chi}|^2)}$$

which can be checked in view of the definition of $\overline{tr\chi}$ and (2.1).

Integrate (2.47) in t in view of $r\overline{\mathrm{tr}\chi} - 2 \to 0$ as $t \to 0$,

$$|\overline{\mathrm{tr}\chi} - \frac{2}{r}| \lesssim \frac{1}{r}\int_0^{s(t)} \{\left|(an)^{-1}\frac{r}{2}\overline{an\mathrm{tr}\chi\kappa}\right| + r(an)^{-1}\overline{an|\hat{\chi}|^2}\} ds(t').$$

In view of (2.24), taking L_t^2 with the help of (2.32) yields

$$\|\overline{\mathrm{tr}\chi} - \frac{2}{r}\|_{L_t^2} \lesssim \|r\kappa\mathrm{tr}\chi\|_{L_t^2L_\omega^1} + \|r|\hat{\chi}|^2\|_{L_t^2L_\omega^1}. \tag{2.49}$$

By (2.29) and (2.38), we can obtain

$$\|r\mathrm{tr}\chi\kappa\|_{L_t^2L_\omega^1} \lesssim \|rV\kappa\|_{L_t^2L_\omega^1} + \|2\frac{r}{s}\kappa\|_{L_t^2L_\omega^1} \lesssim (\Delta_0^2+1)\|\kappa\|_{L_t^2L_\omega^1} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.50}$$

By Proposition 2.3, we have

$$\|r|\hat{\chi}|^2\|_{L_t^2L_\omega^1} \lesssim \|r^{1/2}\hat{\chi}\|_{L_t^\infty L_\omega^2}\|r^{1/2}\hat{\chi}\|_{L_t^2L_\omega^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.51}$$

(2.45) then follows by connecting (2.49), (2.50) and (2.51).

Note that it is straightforward to have

$$\mathrm{tr}\chi - \frac{2}{r} = V - \overline{V} + 2\mathcal{O}sc(\frac{1}{s}) + \overline{\mathrm{tr}\chi} - \frac{2}{r}, \tag{2.52}$$

hence

$$\|\mathrm{tr}\chi - \frac{2}{r}\|_{L_t^2L_\omega^2} \lesssim \|V\|_{L^\infty} + \|\mathcal{O}sc(\frac{1}{s})\|_{L_t^2L_\omega^2} + \|\overline{tr\chi} - \frac{2}{r}\|_{L_t^2}$$

which implies (2.46) with the help of (2.29), (2.37) and (2.45). $\square$

Lemma 2.7. *Denote by $\bar{R}$ one of the quantities, R_0, $tr\chi\underline{A}$, $A\cdot\underline{A}$, there holds*

$$\|\bar{R}\|_{L^2(\mathcal{H}_t)} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.53}$$

Proof. The estimate about R_0 can be obtained directly from (1.13). With the help of Proposition 2.3 and BA1

$$\|A\cdot\underline{A}\|_{L^2(\mathcal{H}_t)} \lesssim \|\underline{A}\|_{L_x^2L_t^\infty}\|A\|_{L_\omega^\infty L_t^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.54}$$

By BA1 and Proposition 2.3, we have

$$\|\mathrm{tr}\chi\underline{A}\|_{L^2(\mathcal{H}_t)} \lesssim \|V\cdot\underline{A}\|_{L^2(\mathcal{H}_t)} + \|s^{-1}\underline{A}\|_{L^2(\mathcal{H}_t)} \lesssim \|r^{-1}\underline{A}\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

The estimate thus follows. $\square$

Lemma 2.8. *Let* $\underline{K} = K - \frac{1}{r^2}$*, then*

$$\|\underline{K}\|_{L^2(\mathcal{H}_t)} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Proof. In view of (2.18) and (1.8),

$$K - \frac{1}{r^2} = \frac{a^2-1}{r^2} + \frac{a^2\iota}{2r} + \frac{\iota(V+\frac{2}{s})a^2}{4} - a\mathrm{tr}\chi(\lambda + \frac{1}{2}\delta) - \check{\rho} \tag{2.55}$$

By $\nabla\!\!\!/_L a = -\nu$, (2.32) and (1.13)

$$\left\|\frac{a^2-1}{r^2}\right\|_{L^2(\mathcal{H}_t)} = \left\|r^{-1}\int_0^s 2a\nu\right\|_{L^2_\omega L^2_t} \lesssim \|r^{-1}\nu\!\!\!/\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.56}$$

In view of (2.55), by (2.29) and $r \approx s$ in (2.22), also using (2.56), (2.46), (2.53),

$$\left\|K - \frac{1}{r^2}\right\|_{L^2(\mathcal{H}_t)} \lesssim \left\|\frac{a^2-1}{r^2}\right\|_{L^2} + \|r^{-1}\iota\|_{L^2} + \|\bar{R}\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0$$

which is the desired estimate. □

With Lemma 2.8, we can prove the following estimates.

Lemma 2.9. *Let* $\mathcal{D}_1$ *be the operator that takes any* S *tangent 1-form* F *to* $(divF,\ curl F)$*. Let* $\mathcal{D}_2$ *be the operator that takes any* S*-tangent symmetric, traceless, 2-tenorfields* F *to* $divF$*. Denote by* $\mathcal{D}$ *one of the operators* $\mathcal{D}_1, \mathcal{D}_2$*. For any appropriate* S *tangent tensor fields* F *in the domain of* $\mathcal{D}$*, if* $\|r^{-\frac{1}{2}}F\|_{L_t^\infty L_x^2} \lesssim \Delta_0$*, there holds*

$$\|\nabla\!\!\!/ F\|_{L^2(\mathcal{H}_t)} + \|r^{-1}F\|_{L^2(\mathcal{H}_t)} \lesssim \|\mathcal{D}F\|_{L^2(\mathcal{H}_t)} + \Delta_0^2 + \mathcal{R}_0. \tag{2.57}$$

For smooth scalar functions Ω*, if* $\|r^{-\frac{1}{2}}\nabla\!\!\!/\Omega\|_{L_t^\infty L_x^2} \lesssim \Delta_0$*, then*

$$\|\nabla\!\!\!/^2\Omega\|_{L^2(\mathcal{H}_t)} + \|r^{-1}\nabla\!\!\!/\Omega\|_{L^2(\mathcal{H}_t)} \lesssim \|\Delta\!\!\!/\Omega\|_{L^2(\mathcal{H}_t)} + \Delta_0^2 + \mathcal{R}_0. \tag{2.58}$$

Proof. Let us prove (2.58) first. In view of Böchner identity on S,

$$\int_S |\nabla\!\!\!/^2\Omega|^2 + K|\nabla\!\!\!/\Omega|^2 = \int_S |\Delta\!\!\!/\Omega|^2, \tag{2.59}$$

we obtain

$$\int_{S_{t'}} |\nabla\!\!\!/^2\Omega|^2 + r^{-2}|\nabla\!\!\!/\Omega|^2 = \int_{S_{t'}} |\Delta\!\!\!/\Omega|^2 - \int_{S_{t'}} \underline{K}|\nabla\!\!\!/\Omega|^2. \tag{2.60}$$

Noticing that on $S_{t'}$, there holds by (**Sob**) and $\|r^{-\frac{1}{2}}\nabla\!\!\!/\Omega\|_{L_t^\infty L_x^2} \lesssim \Delta_0$,

$$\begin{aligned}\|\nabla\!\!\!/\Omega\|^2_{L_x^4(S_{t'})} &\lesssim \|\nabla\!\!\!/^2\Omega\|_{L_x^2(S_{t'})}\|\nabla\!\!\!/\Omega\|_{L_x^2(S_{t'})} + r^{-1}\|\nabla\!\!\!/\Omega\|^2_{L_x^2(S_{t'})}\\ &\lesssim \Delta_0\|\nabla\!\!\!/^2\Omega\|_{L_x^2(S_{t'})} + \Delta_0^2.\end{aligned} \tag{2.61}$$

Integrating (2.60) on $0 < t' \le t$, in view of (2.61) and Lemma 2.8, (2.58) follows by using Young's inequality.

For $\mathcal{D}_1 : F \to (\mathrm{div}F, \mathrm{curl}F)$ and $\mathcal{D}_2 : F \to \mathrm{div}F$ we recall the identities (see [1, Proposition 2.2.1])

$$\int_S |\nabla\!\!\!/ F|^2 + K|F|^2 = \int_S |\mathcal{D}_1F|^2, \quad \int_S |\nabla\!\!\!/ F|^2 + 2K|F|^2 = 2\int_S |\mathcal{D}_2F|^2.$$

Then (2.57) follows in the same way as (2.58). □

By Lemma 2.9 and (2.53), we can derive the following

Lemma 2.10. *For $M = \nabla\!\!\!/ tr\chi, \mu$, there holds*

$$\|\mathcal{D}_0 M\|_{L^2(\mathcal{H}_t)} \lesssim \Delta_0^2 + \mathcal{R}_0 + \|M\|_{L^2(\mathcal{H}_t)}. \tag{2.62}$$

More precisely,

$$\|\nabla\!\!\!/ \hat{\chi}\|_{L^2(\mathcal{H}_t)} \lesssim \Delta_0^2 + \mathcal{R}_0 + \|\nabla\!\!\!/ tr\chi\|_{L^2(\mathcal{H}_t)},$$
$$\|\nabla\!\!\!/ \zeta\|_{L^2(\mathcal{H}_t)} \lesssim \Delta_0^2 + \mathcal{R}_0 + \|\mu\|_{L^2(\mathcal{H}_t)}.$$

Proof. By using $\|r^{-\frac{1}{2}}\underline{A}\|_{L_x^2 L_t^\infty} \lesssim \Delta_0^2 + \mathcal{R}_0$ in Proposition 2.3 and applying Lemma 2.9 to $F = \hat{\chi}, \zeta$, we can derive that

$$\|\nabla\!\!\!/ \hat{\chi}\|_{L^2(\mathcal{H}_t)} \lesssim \|\mathcal{D}_2 \hat{\chi}\|_{L^2(\mathcal{H}_t)} + \Delta_0^2 + \mathcal{R}_0,$$
$$\|\nabla\!\!\!/ \zeta\|_{L^2(\mathcal{H}_t)} \lesssim \|\mathcal{D}_1 \zeta\|_{L^2(\mathcal{H}_t)} + \Delta_0^2 + \mathcal{R}_0.$$

In view of (2.6), (2.8) and (2.9),

$$\mathcal{D}_2 \hat{\chi} = \nabla\!\!\!/ \mathrm{tr}\chi + \bar{R}, \qquad \mathcal{D}_1 \zeta = (\mu, 0) + \bar{R}.$$

By using (2.53), (2.62) can be proved. $\square$

Recall that $M = \nabla\!\!\!/ \mathrm{tr}\chi$ or μ. (2.4) and (2.15) can be symbolically [3] recast as

$$\nabla\!\!\!/_L M + \frac{p}{2}\mathrm{tr}\chi M = \hat{\chi} \cdot M + H_1 + H_2 + H_3 \tag{2.63}$$

where [4]

$$H_1 = A \cdot (\mathcal{D}_0 M + F), \quad H_2 = \underline{A} \cdot A \cdot A, \quad \text{and} \quad H_3 = r^{-1}\bar{R}, \iota \cdot \bar{R},$$

with

$$(p, F) = \begin{cases} (2, \nabla\!\!\!/ \mathrm{tr}\chi) & \text{if } M = \mu \\ (3, 0) & \text{if } M = \nabla\!\!\!/ \mathrm{tr}\chi. \end{cases} \tag{2.64}$$

We will establish the following estimates

Proposition 2.4. *Let M denote either $\nabla\!\!\!/ tr\chi$ or μ, there hold*

$$\|r^{1/2} M\|_{L_x^2 L_t^\infty} + \|M\|_{L^2} \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{2.65}$$

$$\|\nabla\!\!\!/ \zeta, \nabla\!\!\!/ \hat{\chi}\|_{L^2} \lesssim \mathcal{R}_0 + \Delta_0^2. \tag{2.66}$$

Proof. Noticing that (2.66) can be obtained immediately by combining the second estimate in (2.65) with Lemma 2.10, we first consider the second norm in (2.65). By (2.31), (2.63) and Lemma 2.1, integrating along the null geodesic Γ_ω initiating from vertex, we obtain

$$|M| \leq \sum_{i=1}^{3} \left| v_t^{-\frac{p}{2}} \int_0^t v_{t'}^{\frac{p}{2}} |H_i| n a dt' \right| = I_1(t) + I_2(t) + I_3(t). \tag{2.67}$$

[3]This means signs and coefficients on the right side of the expression can be ignored.

[4]In view of $|a-1| \leq 1/2$ in BA1, the factors a^m, $m \in \mathbb{N}$ can be ignored in H_i when we employ (2.63) to prove Proposition 2.4.

Consider H_3 with the help of (2.32),

$$\int_0^1 \left|v_t^{1/2} I_3(t)\right|^2 nadt = \int_0^1 \left(v_t^{-\frac{p}{2}+\frac{1}{2}} \int_0^t v_{t'}^{\frac{p}{2}} |H_3| nadt'\right)^2 nadt \tag{2.68}$$
$$\lesssim \int_0^1 \left|r^{-p+1} \int_0^t r'^{p-2} |r' \bar{R}| nadt'\right|^2 nadt$$
$$\lesssim \left|\int_0^1 |r' \bar{R}|^2 nadt'\right|$$

where we employed $v_{t'} \approx (r')^2 \approx (t')^2$.

Then by taking $L^1_\omega(\mathbb{S}^2)$, with the help of (2.53), we obtain

$$\|I_3\|_{L^2} \lesssim \|\bar{R}\|_{L^2} \lesssim \mathcal{R}_0 + \Delta_0^2. \tag{2.69}$$

Similarly, we have

$$\int_0^1 \left|v_t^{1/2} I_2(t)\right|^2 nadt = \int_0^1 \left(v_t^{-\frac{p}{2}+\frac{1}{2}} \int_0^t v_{t'}^{\frac{p}{2}} |H_2| nadt'\right)^2 nadt$$
$$\le \int_0^1 \left(v_t^{-\frac{p}{2}+\frac{1}{2}} \int_0^t v_{t'}^{\frac{p}{2}} |A|^2 |\underline{A}| nadt'\right)^2 nadt$$
$$\lesssim \|A\|^4_{L^2_t} \|r^{1/2} \underline{A}\|^2_{L^\infty_t}.$$

By taking L^1_ω, we obtain in view of BA1 and Proposition 2.3 that

$$\|I_2\|_{L^2} \lesssim \|A\|^2_{L^\infty_\omega L^2_t} \|r^{1/2} \underline{A}\|_{L^2_\omega L^\infty_t} \lesssim \Delta_0^2(\Delta_0^2 + \mathcal{R}_0). \tag{2.70}$$

Now consider $H_1 := A \cdot (\mathcal{D}_0 M + F)$. We have

$$\left|v_t^{1/2} I_1(t)\right| \lesssim \|A\|_{L^2_t} \left(v_t^{-p+1} \int_0^{s(t)} v_{t'}^p \left(|\mathcal{D}_0 M|^2 + |F|^2\right) ds(t')\right)^{1/2}. \tag{2.71}$$

Take L^2_ω,

$$\|v_t^{1/2} I_1\|_{L^2_\omega} \lesssim \|A\|_{L^\infty_\omega L^2_t} (\|\mathcal{D}_0 M\|_{L^2} + \|F\|_{L^2}).$$

Hence in view of (2.62) and BA1

$$\|I_1\|_{L^2} \lesssim \Delta_0(\|M\|_{L^2} + \|F\|_{L^2} + \Delta_0^2 + \mathcal{R}_0). \tag{2.72}$$

Consequently, in view of (2.69), (2.70) and (2.72), if $M = \nabla\!\!\!/ \mathrm{tr}\chi$, we have

$$\|\nabla\!\!\!/ \mathrm{tr}\chi\|_{L^2} \lesssim \Delta_0 \|\nabla\!\!\!/ \mathrm{tr}\chi\|_{L^2} + \Delta_0^2 + \mathcal{R}_0, \tag{2.73}$$

and if $M = \mu$

$$\|\mu\|_{L^2} \lesssim \Delta_0(\|\mu\|_{L^2} + \|\nabla\!\!\!/ \mathrm{tr}\chi\|_{L^2}) + \Delta_0^2 + \mathcal{R}_0. \tag{2.74}$$

From (2.73), noticing that $0 < \Delta_0 < 1/2$, we conclude

$$\|\nabla\!\!\!/ \mathrm{tr}\chi\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.75}$$

$\|\mu\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0$ then follows from (2.74) by using (2.75).

Now we consider the first estimate in (2.65).

$$v_t^{\frac{3}{4}} I_3(t) \le v_t^{\frac{3}{4}-\frac{p}{2}} \int_0^{s(t)} v_{t'}^{\frac{p}{2}-\frac{1}{2}} |\bar{R}| ds(t') \lesssim s^{\frac{3}{2}-p} \int_0^{s(t)} s^{p-2} |\bar{R} r| ds(t')$$
$$\lesssim \left(\int_0^{s(t)} |\bar{R} r|^2 ds(t') \right)^{\frac{1}{2}}$$

where we used $s^2 \approx v_t$, $r \approx s$ in Proposition 2.2 to derive the second inequality. Thus

$$\left\| \sup_{t\in(0,1]} |v_t^{\frac{3}{4}} I_3(t)| \right\|_{L^2_\omega} \lesssim \|\bar{R}\|_{L^2(\mathcal{H})} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.76}$$

We can proceed in a similar fashion for the other two terms,

$$v_t^{\frac{3}{4}} I_2(t) \le v_t^{\frac{3}{4}-\frac{p}{2}} \int_0^{s(t)} v_{t'}^{\frac{p}{2}} |A|^2 |\underline{A}| ds(t') \lesssim s^{\frac{3}{2}-p} \int_0^{s(t)} s^p |A|^2 |\underline{A}| ds(t')$$
$$\lesssim r \|A\|^2_{L^2_t} \|r^{1/2} \underline{A}\|_{L^\infty_t},$$

then

$$\left\| \sup_{0<t\le 1} |v_t^{\frac{3}{4}} I_2(t)| \right\|_{L^2_\omega} \lesssim \|A\|^2_{L^\infty_\omega L^2_t} \|r^{1/2} \underline{A}\|_{L^2_\omega L^\infty_t} \lesssim \Delta_0^2(\Delta_0^2 + \mathcal{R}_0). \tag{2.77}$$

Similarly,

$$v_t^{\frac{3}{4}} I_1(t) \lesssim \|A\|_{L^2_t} \|r(|\mathcal{D}_0 M| + |F|)\|_{L^2_t} r^{1/2}.$$

Taking L^2_ω, using (2.66) for $\mathcal{D}_0 M$ and (2.75) for F, also in view of BA1

$$\left\| \sup_{0<t\le 1} |v_t^{\frac{3}{4}} I_1(t)| \right\|_{L^2_\omega} \lesssim (\|\mathcal{D}_0 M\|_{L^2} + \|F\|_{L^2}) \|A\|_{L^\infty_\omega L^2_t} \lesssim (\Delta_0^2 + \mathcal{R}_0)\Delta_0. \tag{2.78}$$

Combine (2.76), (2.77), (2.78) we conclude that $\|r^{1/2} M\|_{L^2_x L^\infty_t(\mathcal{H})} \lesssim \Delta_0^2 + \mathcal{R}_0$. □

Let us summarize the major estimates that have been obtained so far.

Proposition 2.5. *Let M denote either μ or $\nabla\!\!\!\!/ \, tr\chi$. There holds*

$$\mathcal{N}_1(\underline{A}) + \|r^{1/2} M\|_{L^2_x L^\infty_t} + \|M\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0 \tag{2.79}$$

Remark 2.1. By (**SobM1**),(**Sob**) and (1.13), it is easy to check $(a^m A)$ and $(a^m \underline{A})$ with $m \in \mathbb{N}$ verify the same estimates as A and $\underline{A}$ respectively. Consequently they can also be regarded as elements of A and $\underline{A}$ respectively.

2.4. **More Estimates for κ and ι.** The main purpose of this subsection is to provide estimates for $\mathcal{N}_1(\kappa, \iota)$ and $\|\kappa, \iota\|_{L^2_t L^\infty_\omega}$. We first derive a simple consequence from (2.79) with the help of (2.26).

Lemma 2.11.

$$\|r^{-1} \mathcal{O}sc(an)\|_{L^2_t L^\infty_\omega} \lesssim \Delta_0^2 + \mathcal{R}_0 \tag{2.80}$$

Proof. Apply (2.26) to $\Omega = an - \overline{an}$,

$$\|r^{-1}\Omega\|_{L_t^2 L_\omega^\infty} \lesssim \|\nabla\!\!\!/^2\Omega\|_{L^2} + \|r^{-1}\nabla\!\!\!/\Omega\|_{L^2}. \tag{2.81}$$

Note that with the help of $\zeta + \underline{\zeta} = \nabla\!\!\!/ \log(an)$ and (2.24), there holds

$$\|r^{-\frac{1}{2}}\nabla\!\!\!/\Omega\|_{L_t^\infty L_x^2} \lesssim \|r^{-\frac{1}{2}}\nabla\!\!\!/ \log(an)\|_{L_t^\infty L_x^2} = \|r^{-\frac{1}{2}}(\zeta+\underline{\zeta})\|_{L_t^\infty L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

in view of (2.58), we deduce

$$\|r^{-1}\Omega\|_{L_t^2 L_\omega^\infty} \lesssim \|\Delta\!\!\!/\Omega\|_{L^2} + \Delta_0^2 + \mathcal{R}_0.$$

We obtain in view of $\Delta\!\!\!/(an) = an(|\zeta+\underline{\zeta}|^2 + \Delta\!\!\!/ \log(an))$, (2.24)

$$\begin{aligned}
\|r^{-1}\Omega\|_{L_t^2 L_\omega^\infty} &\lesssim \|\Delta\!\!\!/ \log(an)\|_{L^2} + \|\underline{\zeta}+\zeta\|_{L^4}^2 + \Delta_0^2 + \mathcal{R}_0\\
&\lesssim \mathcal{N}_1(\zeta+\underline{\zeta})(1+\mathcal{N}_1(\zeta+\underline{\zeta})) + \Delta_0^2 + \mathcal{R}_0\\
&\lesssim \Delta_0^2 + \mathcal{R}_0
\end{aligned} \tag{2.82}$$

where we employed (**SobM1**) and (2.79) for the last two inequalities. □

Proposition 2.6.

$$\|r\nabla\!\!\!/^2(\frac{1}{s})\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{2.83}$$

$$\|\mathcal{O}sc(\frac{1}{s}), \mathcal{O}sc(tr\chi), \kappa, \iota\|_{L_t^2 L_\omega^\infty} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.84}$$

Proof. Using (2.39), in view of the commutation formula in [1, Lemma 13.1.2], symbolically, we obtain

$$\begin{aligned}
\frac{d}{ds}\nabla\!\!\!/^2 s + \mathrm{tr}\chi\nabla\!\!\!/^2 s = &-\frac{1}{2}\nabla\!\!\!/\mathrm{tr}\chi\nabla\!\!\!/ s + \hat{\chi}\cdot\nabla\!\!\!/^2 s - \frac{1}{2}\mathrm{tr}\chi(\zeta+\underline{\zeta})\nabla\!\!\!/ s - \nabla\!\!\!/\hat{\chi}\cdot\nabla\!\!\!/ s\\
&-(\zeta+\underline{\zeta})\hat{\chi}\cdot\nabla\!\!\!/ s + (\zeta+\underline{\zeta})\cdot(\zeta+\underline{\zeta}) + \nabla\!\!\!/(\zeta+\underline{\zeta}) + (\chi\cdot\underline{\zeta}+\beta)\nabla\!\!\!/ s.
\end{aligned}$$

We then rewrite it as

$$\frac{d}{ds}\nabla\!\!\!/^2 s + \mathrm{tr}\chi\nabla\!\!\!/^2 s = \hat{\chi}\cdot\nabla\!\!\!/^2 s + (\bar{R} + M + \nabla\!\!\!/\hat{\chi})\cdot\nabla\!\!\!/ s + A\cdot A + \nabla\!\!\!/(\zeta+\underline{\zeta}).$$

Apply Lemma 2.3 to the above equation with the help of $\|\hat{\chi}\|_{L_\omega^\infty L_t^2(\mathcal{H})} \le \Delta_0$ in BA1 and $\lim_{t\to 0} r^2\nabla\!\!\!/^2 s = 0$,

$$|\nabla\!\!\!/^2 s| \lesssim v_t^{-1}\int_0^{s(t)} v_{t'}|(\bar{R}+M+\nabla\!\!\!/\hat{\chi})\cdot\nabla\!\!\!/ s + A\cdot A + \nabla\!\!\!/(\zeta+\underline{\zeta})|ds(t'). \tag{2.85}$$

Hence, by Hölder inequality and (2.32), we have

$$\begin{aligned}
\|\nabla\!\!\!/^2 s\|_{L_\omega^2 L_t^2} \lesssim \|s^{-1}\nabla\!\!\!/ s\|_{L_\omega^\infty L_t^2}(\|\bar{R}\|_{L^2} + \|M\|_{L^2} + \|\nabla\!\!\!/ A\|_{L^2})\\
+ \|r(A\cdot A + \nabla\!\!\!/ A)\|_{L_\omega^2 L_t^2}.
\end{aligned} \tag{2.86}$$

By (2.53), (2.65), (2.35), (2.54) and (2.79), we obtain

$$\|\nabla\!\!\!/^2 s\|_{L_\omega^2 L_t^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.87}$$

By a straightforward calculation,

$$-s^2\nabla\!\!\!/^2(\frac{1}{s}) = s^2\nabla\!\!\!/(s^{-2}\nabla\!\!\!/ s) = \nabla\!\!\!/^2 s - 2s^{-1}\nabla\!\!\!/ s\cdot\nabla\!\!\!/ s$$

we deduce

$$\|r\nabla\!\!\!/^2(\mathcal{O}sc(1/s))\|_{L^2} \lesssim \|\nabla\!\!\!/^2 s\|_{L_t^2 L_\omega^2} + \|\nabla\!\!\!/ \log s\cdot\nabla\!\!\!/ s\|_{L_t^2 L_\omega^2}.$$

By (2.36) and (2.35)

$$\|\nabla\!\!\!/ \log s \cdot \nabla\!\!\!/ s\|_{L_t^2 L_\omega^2} \lesssim \|\nabla\!\!\!/ \log s\|_{L_\omega^\infty L_t^2} \|s \nabla\!\!\!/ \log s\|_{L_\omega^2 L_t^\infty} \lesssim \Delta_0(\Delta_0^2 + \mathcal{R}_0). \tag{2.88}$$

Combined with (2.87),

$$\|r \nabla\!\!\!/^2 (\mathcal{O}sc(1/s))\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.89}$$

Now we apply (2.26) to $\Omega = r\mathcal{O}sc(\frac{1}{s})$. By (2.36), $\|\nabla\!\!\!/ (\mathcal{O}sc(\frac{1}{s}))\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0$. Also in view of (2.89), we conclude that

$$\|\mathcal{O}sc(\frac{1}{s})\|_{L_t^2 L_\omega^\infty} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.90}$$

By $\mathcal{O}sc(\mathrm{tr}\chi) = \mathcal{O}sc(V) + 2\mathcal{O}sc(\frac{1}{s})$ and (2.29), it follows from (2.90) that

$$\|\mathcal{O}sc(\mathrm{tr}\chi)\|_{L_t^2 L_\omega^\infty} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{2.91}$$

In view of $\iota = \mathcal{O}sc(\mathrm{tr}\chi) + \overline{\mathrm{tr}\chi} - \frac{2}{r}$, (2.91) together with (2.45) implies

$$\|\iota\|_{L_t^2 L_\omega^\infty} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

In view of (2.41), symbolically,

$$\kappa = V - (an)^{-1}\overline{anV} + \mathcal{O}sc(\frac{1}{s})\frac{\overline{an}}{an} + s^{-1}(an)^{-1}\mathcal{O}sc(an) + (an)^{-1}\overline{s^{-1}\mathcal{O}sc(an)}. \tag{2.92}$$

Taking $L_t^2 L_\omega^\infty$ of κ in view of (2.92), by (2.80) and (2.24), the last two terms are bounded by $\Delta_0^2 + \mathcal{R}_0$. Due to (2.90) and (2.29), $L_t^2 L_\omega^\infty$ of the remaining three terms are bounded by $\Delta_0^2 + \mathcal{R}_0$. Thus we conclude $\|\kappa\|_{L_t^2 L_\omega^\infty} \lesssim \Delta_0^2 + \mathcal{R}_0$. □

Proposition 2.7.

$$\mathcal{N}_1(\iota) + \mathcal{N}_1(\kappa) \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Proof. In view of (2.1) and (2.48), we have

$$\nabla\!\!\!/_L(\mathrm{tr}\chi - \frac{2}{r}) = -\frac{1}{2}(\mathrm{tr}\chi - \frac{2}{r})\mathrm{tr}\chi - |\hat{\chi}|^2 - \frac{1}{r}\kappa.$$

By BA1 and (2.46), (2.38) and (2.54), we have $\|\nabla\!\!\!/_L(\iota)\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0$. Together with (2.46) and (2.79), we conclude $\mathcal{N}_1(\iota) \lesssim \Delta_0^2 + \mathcal{R}_0$.

Now we take $L^2(\mathcal{H})$ norm of $\frac{d}{ds}\kappa$ with the help of

$$\begin{aligned} \frac{d}{ds}\kappa + \mathrm{tr}\chi \cdot \kappa = &-|\hat{\chi}|^2 + (an)^{-2}\overline{|an\hat{\chi}|^2} + \frac{1}{2}\kappa^2 - \frac{1}{2}(an)^{-2}\overline{(an)^2\kappa^2} \\ &+ (an)^{-2}\left(\overline{an\mathrm{tr}\chi}\mathcal{O}sc(\nabla\!\!\!/_L(an)) - \overline{\mathcal{O}sc(\nabla\!\!\!/_L(an))an\kappa}\right). \end{aligned} \tag{2.93}$$

By (2.38) and (2.23), $\|\mathrm{tr}\chi\kappa\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0$. For the terms on the right of (2.93), we first claim

$$\|r^{-1}\mathcal{O}sc(\nabla\!\!\!/_L(an))\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0 \tag{2.94}$$

Indeed,

$$\nabla\!\!\!/ \nabla\!\!\!/_L(an) = \nabla\!\!\!/(an\nabla\!\!\!/_L \log(an)) = \nabla\!\!\!/ \mathcal{D}_t \log(an)$$

by (2.21),

$$\nabla\!\!\!/ \nabla\!\!\!/_L(an) = \mathcal{D}_t \nabla\!\!\!/ \log(an) + an\chi \cdot \nabla\!\!\!/ \log(an) = an\{\nabla\!\!\!/_L(\zeta + \underline{\zeta}) + \chi \cdot (\zeta + \underline{\zeta})\}. \tag{2.95}$$

By (**Poin**), (2.95) and (2.24)

$$\begin{aligned} \|r^{-1}\mathcal{O}sc(\nabla\!\!\!/_L(an))\|_{L^2} &\lesssim \|\nabla\!\!\!/ \nabla\!\!\!/_L(an)\|_{L^2} \\ &\lesssim \|\mathrm{tr}\chi A\|_{L^2} + \|\nabla\!\!\!/_L A\|_{L^2} + \|\hat{\chi} \cdot A\|_{L^2} \end{aligned} \tag{2.96}$$

(2.94) follows by using (2.53) and Proposition 2.3.

Using (2.94), (2.24) and (2.23)

$$\|(an)^{-2}\overline{an\mathrm{tr}\chi}\mathcal{O}sc(\nabla\!\!\!/_L(an))\|_{L^2}\lesssim\|r^{-1}\mathcal{O}sc(\nabla\!\!\!/_L(an))\|_{L^2}\lesssim\Delta_0^2+\mathcal{R}_0.$$

Similarly

$$\|(an)^{-2}\overline{\mathcal{O}sc(\nabla\!\!\!/_L(an))an\kappa}\|_{L^2}\lesssim\|\mathcal{O}sc(\nabla\!\!\!/_L(an))\|_{L_t^2L_\omega^2}\lesssim\Delta_0^2+\mathcal{R}_0.$$

Now consider other terms on the right of (2.93), by (2.38) and (2.84)

$$\|\kappa^2\|_{L^2}\lesssim\|r\kappa\|_{L_t^\infty L_\omega^2}\|\kappa\|_{L_t^2L_\omega^\infty}\lesssim\Delta_0^2+\mathcal{R}_0,$$

by (**SobM1**) and (2.79)

$$\|\hat{\chi}\cdot\hat{\chi}\|_{L^2}\lesssim\|\hat{\chi}\|_{L^4}^2\lesssim\Delta_0^2+\mathcal{R}_0$$

and the other two terms can be estimated similarly in view of (2.24). Hence

$$\|\nabla\!\!\!/_L\kappa\|_{L^2}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{2.97}$$

At last, we obtain by definition

$$\nabla\!\!\!/\kappa=\nabla\!\!\!/\mathrm{tr}\chi+\nabla\!\!\!/\log(an)\cdot(an)^{-1}\overline{an\mathrm{tr}\chi}.$$

By using(2.23), (2.24) and (2.79),

$$\|\nabla\!\!\!/\kappa\|_{L^2(\mathcal{H})}\lesssim\|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(\mathcal{H})}+\|r^{-1}(\zeta+\underline{\zeta})\|_{L^2(\mathcal{H})}\lesssim\Delta_0^2+\mathcal{R}_0.$$

Combined with (2.38) and (2.97), we can conclude $\mathcal{N}_1(\kappa)\lesssim\Delta_0^2+\mathcal{R}_0$. □

Remark 2.2. By Proposition 2.7 and (2.84), we can regard κ and ι as elements of A.

Without making the strong assumption (1.23) (see [7]), under the weaker condition (1.13) only, $\mathrm{tr}\chi-\frac{2}{r}$ no longer satisfies the $L^\infty(\mathcal{H})$ estimate as (1.14) for $\mathrm{tr}\chi-\frac{2}{s}$. Since S_t is not a level set of affine parameter s, obviously, $\nabla\!\!\!/ r=0$ while $\nabla\!\!\!/ s\neq 0$. We will check the weakly spherical property for $\overset{\circ}{\gamma}=r^{-2}\gamma$. Integral operators $\Lambda^{-\alpha}$, geometric Littlewood Paley decompositions P_k and Besov norms will then be defined by heat flow $U(\tau)$ with respect to $\overset{\circ}{\gamma}$ instead of $s^{-2}\gamma$. Due to the factor "an" in (2.48), the nontrivial evolution of r adds technical complexity. This issue can be settled by proving $\mathrm{tr}\chi-(an)^{-1}\overline{an\mathrm{tr}\chi}$ and $\mathrm{tr}\chi-\frac{2}{r}$ verify stronger estimates than $\hat{\chi}$. Examples of the application of (2.84) and Proposition 2.7 can be seen in the proof of (2.100), Proposition 3.3, Theorem 5.1, etc.

2.5. **Weakly spherical surfaces.** Let γ be the restriction metric on S_t, and define the rescaled metric $\overset{\circ}{\gamma}$ on S_t by $\overset{\circ}{\gamma}=r^{-2}\gamma$. Let $\gamma_{ij}^{(0)}$ denote the canonical metric on $\mathbb{S}^2$. Note that

$$\lim_{t\to 0}\overset{\circ}{\gamma}_{ij}=\gamma_{ij}^{(0)},\qquad \lim_{t\to 0}\partial_k\overset{\circ}{\gamma}_{ij}=\partial_k\gamma_{ij}^{(0)} \tag{2.98}$$

where $i,j,k=1,2$. With its aid, using the bootstrap assumption BA1, (2.79), we will prove that for each $0<t\le 1$ the leave $(S_t,\overset{\circ}{\gamma})$ is a weakly spherical surface.

Proposition 2.8. *For the transport local coordinates* (t,ω), *the following properties hold true for all surfaces* S_t *of the time foliation on the null cone* $\mathcal{H}$*: the metric* $\overset{\circ}{\gamma}_{ij}(t)$ *on each* S_t *verifies weakly spherical conditions i.e.*

$$\|\overset{\circ}{\gamma}_{ij}(t)-\gamma_{ij}^{(0)}\|_{L_\omega^\infty}\lesssim\Delta_0 \tag{2.99}$$

$$\|\partial_k\overset{\circ}{\gamma}_{ij}(t)-\partial_k\gamma_{ij}^{(0)}\|_{L_\omega^2L_t^\infty}\lesssim\Delta_0 \tag{2.100}$$

Proof. Since relative to the transport coordinate on $\mathcal{H}$, $\frac{d}{ds}\gamma_{ij}=2\chi_{ij}$,

$$\frac{d}{dt}(\overset{\circ}{\gamma}_{ij})=an(\kappa\cdot\gamma_{ij}+2\hat{\chi}_{ij})r^{-2}. \tag{2.101}$$

Integrating (2.101) along null geodesic initiating from vertex, with the help of (2.98), (2.99) follows in view of (2.84) and BA1.

Integrating the following transport equation along a null geodesic initiating from vertex,

$$\begin{aligned}\frac{d}{dt}\partial_k\overset{\circ}{\gamma}_{ij}=an\Big(&\partial_k\log(an)\kappa\overset{\circ}{\gamma}_{ij}+\partial_k\mathrm{tr}\chi\overset{\circ}{\gamma}_{ij}\\&+\kappa\partial_k\overset{\circ}{\gamma}_{ij}+2\partial_k\hat{\chi}_{ij}r^{-2}+2\partial_k\log(an)\hat{\chi}_{ij}r^{-2}\Big)\end{aligned}$$

where $i,j,k=1,2$, with the initial condition given by (2.98), by $\|\kappa\|_{L_t^2L_\omega^\infty}\lesssim\Delta_0^2+\mathcal{R}_0$ in (2.84) and a similar argument to Lemma 2.3, we can obtain

$$\begin{aligned}\Big|\partial_k\overset{\circ}{\gamma}_{ij}(t)-\partial_k\gamma_{ij}^{(0)}\Big|\lesssim\Big|\int_0^{s(t)}&\Big(\partial_k\mathrm{tr}\chi\cdot\overset{\circ}{\gamma}_{ij}+r^{-2}(\nabla\!\!\!\!/_k\hat{\chi}_{ij}-\Gamma\cdot\hat{\chi})\\&+\partial_k\log(an)\overset{\circ}{\gamma}_{ij}\kappa+2\partial_k\log(an)\hat{\chi}_{ij}r^{-2}+\kappa\partial_k\gamma_{ij}^{(0)}\Big)\,ds(t')\Big|,\end{aligned}$$

where Γ represents Christoffel symbols, and $\Gamma\cdot\hat{\chi}$ stands for the terms $\sum_{l=1}^2\Gamma_{ki}^l\hat{\chi}_{lj}$ with $l=1,2$. Then with the help of (2.99),

$$\begin{aligned}&\Big\|\sup_{0<t\le1}\Big|\partial_k\overset{\circ}{\gamma}_{ij}(t)-\partial_k\gamma_{ij}^{(0)}\Big|\Big\|_{L_\omega^2}\\&\lesssim\|\Gamma\|_{L_\omega^2L_t^2}\|\hat{\chi}\|_{L_\omega^\infty L_t^2}+\|\nabla\!\!\!\!/\mathrm{tr}\chi\|_{L_x^2L_t^1}+\|\nabla\!\!\!\!/\hat{\chi}\|_{L_x^2L_t^1}\\&+\|r(\zeta+\underline{\zeta})\cdot\kappa\|_{L_\omega^2L_t^1}+\big\|r|\zeta+\underline{\zeta}||\hat{\chi}|\big\|_{L_\omega^2L_t^1}+\|\kappa\|_{L_t^2L_\omega^2}.\end{aligned} \tag{2.102}$$

By BA1, (2.38), (2.54), we obtain the terms in the line of

$$(2.102)\lesssim\|\kappa\|_{L_t^2L_\omega^2}(\|A\|_{L_\omega^\infty L_t^2}+1)+\|A\cdot A\|_{L^2(\mathcal{H})}\lesssim\Delta_0^2+\mathcal{R}_0.$$

Using Proposition 2.4,

$$\Big\|\sup_{0<t\le1}\Big|\partial_k\overset{\circ}{\gamma}_{ij}(t)-\partial_k\gamma_{ij}^{(0)}\Big|\Big\|_{L_\omega^2}\lesssim\Delta_0^2+\mathcal{R}_0+\|\hat{\chi}\|_{L_\omega^\infty L_t^2}\|\Gamma\|_{L_\omega^2L_t^2}.$$

Sum over all $i,j,k=1,2$, also using (2.99)

$$\|\Gamma\|_{L_\omega^2}\lesssim\sum_{i,j,k=1,2}\|\partial_k(\overset{\circ}{\gamma}_{ij}-\gamma_{ij}^{(0)})\|_{L_\omega^2}+C,$$

where C is the constant such that the Christoffel symbol of $\gamma^{(0)}$ satisfies $|\partial\gamma^{(0)}|\le C$, (2.100) then follows by using $\|\hat{\chi}\|_{L_\omega^\infty L_t^2}\le\Delta_0<1/2$ in BA1. $\square$

3. $\|\Lambda^{-\alpha}\underline{K}\|_{L_t^\infty L_x^2}$ and elliptic estimates

Define the operator Λ^a with $a \le 0$ such that for any S-tangent tensor fields F

$$\Lambda^a F := \frac{r^{-a}}{\Gamma(-a/2)} \int_0^\infty \tau^{-\frac{a}{2}-1} e^{-\tau} U(\tau) F d\tau, \tag{3.1}$$

where Γ denotes Gamma function and $U(\tau)F$ is defined on $(S_t, \overset{\circ}{\gamma})$ by

$$\frac{\partial}{\partial \tau} U(\tau)F - \Delta_{\overset{\circ}{\gamma}} U(\tau)F = 0, \quad U(0)F = F. \tag{3.2}$$

The definition of Λ^a extends to the range $a > 0$ by defining for $0 < a \le 2m$ that

$$\Lambda^a F = \Lambda^{a-2m} \cdot (r^{-2} Id - \Delta_\gamma)^m F.$$

We record the basic properties of Λ^a in the following result (see [3]).

Proposition 3.1. (i) *$\Lambda^0 = Id$ and $\Lambda^a \cdot \Lambda^b = \Lambda^{a+b}$ for any $a, b \in \mathbb{R}$.*

(ii) *For any S-tangent tensor field F and any $a \le 0$*

$$r^a \|\Lambda^a F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.$$

(iii) *For any S-tangent tensor field F and any $b \ge a \ge 0$*

$$r^a \|\Lambda^a F\|_{L^2(S)} \lesssim r^b \|\Lambda^b F\|_{L^2(S)} \quad \text{and} \quad \|\Lambda^a F\|_{L^2(S)} \lesssim \|\Lambda^b F\|_{L^2(S)}^{\frac{a}{b}} \|F\|_{L^2(S)}^{1-\frac{a}{b}}.$$

(iv) *For any S-tangent tensor fields F and G and any $0 \le a < 1$*

$$\|\Lambda^a (F \cdot G)\|_{L^2(S)} \lesssim \|\Lambda F\|_{L^2(S)} \|\Lambda^a G\|_{L^2(S)} + \|\Lambda^a F\|_{L^2(S)} \|\Lambda G\|_{L^2(S)}.$$

(v) *For any S-tangent tensor field F there holds with $2 < p < \infty$ and $a > 1 - \frac{2}{p}$*

$$\|F\|_{L^p(S)} \lesssim \|\Lambda^a F\|_{L^2(S)}.$$

(vi) *For any $a \in \mathbb{R}$ and any S-tangent tensor field F*

$$\|F\|_{H^a(S)} := \|\Lambda^a F\|_{L^2(S)}$$

Proposition 3.2. *Under the assumption of BA1 , if $\mathcal{R}_0 > 0$ is sufficiently small, then for all $\frac{1}{2} \le \alpha < 1$ there hold*

$$\underline{K}_\alpha := \|\Lambda^{-\alpha}(K - r^{-2})\|_{L_t^\infty L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{3.3}$$

$$\|\Lambda^{-\alpha} \check{\rho}\|_{L_t^\infty L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{3.4}$$

For smooth scalar functions f on $\mathcal{H}$, with $p > 2$, $\alpha \ge 1/2$, define the good part of the commutator $[\Lambda^{-\alpha}, \mathcal{D}_t] f$ by

$$[\Lambda^{-\alpha}, \mathcal{D}_t]_g f := r^\alpha C_\alpha \int_0^\infty \tau^{\frac{\alpha}{2}-1} e^{-\tau} [U(\tau), \mathcal{D}_t] f d\tau, \text{ with } C_\alpha = \frac{1}{\Gamma(\frac{\alpha}{2})}.$$

We first prove Proposition 3.2 by assuming the following result.

Proposition 3.3. *Let f be a smooth scalar function $\mathcal{H}$, there holds*

$$\|r^{-\alpha} [\Lambda^{-\alpha}, \mathcal{D}_t]_g f\|_{L_t^1 L_x^2} \lesssim (1 + I_\alpha^{1-\frac{2}{p}})(\Delta_0^2 + \mathcal{R}_0) \|f\|_{L^2},$$

where $p > 2$ and $1/2 \le \alpha < 1$, and $I_\alpha := 1 + \underline{K}_\alpha^{\frac{1}{1-\alpha}} + \underline{K}_\alpha^{\frac{1}{2}}$.

Proof of Proposition 3.2. It suffices to prove the following estimates

$$\|\Lambda^{-\alpha}(\underline{K}+\check{\rho})\|_{L_t^\infty L_x^2} \lesssim \Delta_0^2+\mathcal{R}_0, \tag{3.5}$$

$$\|\Lambda^{-\alpha}\check{\rho}\|_{L_t^\infty L_x^2} \lesssim (\Delta_0^2+\mathcal{R}_0)(1+I_\alpha^{1-\frac{2}{p}}), \quad p>2. \tag{3.6}$$

By (3.5) and (3.6), according to the definition of I_α, with $\frac{1}{1-\alpha}\cdot(1-\frac{2}{p})<1$, we can obtain (3.3).

Let us first prove (3.5). By (2.55),

$$\underline{K}+\check{\rho}=\frac{a^2-1}{r^2}+\frac{a^2\iota}{2r}+\frac{\iota\mathrm{tr}\chi a^2}{4}+\mathrm{tr}\chi\underline{A}, \tag{3.7}$$

By Proposition 3.1 (ii), BA1, (2.22) Proposition 2.3 and (2.23), for $\alpha\geq 1/2$,

$$\|\Lambda^{-\alpha}(\mathrm{tr}\chi\underline{A})\|_{L_t^\infty L_x^2}\lesssim\|r^{\alpha+1}\mathrm{tr}\chi\underline{A}\|_{L_t^\infty L_\omega^2}\lesssim\Delta_0^2+\mathcal{R}_0.$$

By Proposition 3.1 (ii) and (1.13), we have for $\alpha\geq\frac{1}{2}$

$$\begin{aligned}\left\|\Lambda^{-\alpha}(\frac{a^2-1}{r^2})\right\|_{L_t^\infty L_x^2}&\lesssim\|r^{-1}\Lambda^{-\alpha}(a^2-1)\|_{L_t^\infty L_\omega^2}\lesssim\|r^{\alpha-1}(a^2-1)\|_{L_t^\infty L_\omega^2}\\&\lesssim\left\|r^{\alpha-1}\int_0^t a^2 n\nabla\!\!\!/_L a dt'\right\|_{L_t^\infty L_\omega^2}\lesssim\|\nabla\!\!\!/_L a\|_{L_\omega^2 L_t^2}\\&\lesssim\|r^{-1}\nu\|_{L^2}\lesssim\Delta_0^2+\mathcal{R}_0.\end{aligned}$$

By Proposition 3.1 (ii), (**SobM1**) and Proposition 2.7, we have for $\alpha\geq\frac{1}{2}$,

$$\left\|\Lambda^{-\alpha}(\frac{a^2\iota}{r})\right\|_{L_t^\infty L_x^2}\lesssim\|a^2r^\alpha\iota\|_{L_t^\infty L_\omega^2}\lesssim\mathcal{N}_1(\iota)\lesssim\Delta_0^2+\mathcal{R}_0.$$

Similarly, by (2.23),

$$\|\Lambda^{-\alpha}(a^2\mathrm{tr}\chi\iota)\|_{L_t^\infty L_x^2}\lesssim\|r^{1+\alpha}(a^2\mathrm{tr}\chi\iota)\|_{L_t^\infty L_\omega^2}\lesssim\|r^\alpha\iota\|_{L_t^\infty L_\omega^2}\lesssim\Delta_0^2+\mathcal{R}_0.$$

This finishes the proof of (3.5).

Next we prove (3.6). Let $W(t)=(\Lambda^{-\alpha}\check{\rho})^2(t)-(\Lambda^{-\alpha}\check{\rho})^2(0)$. Then

$$\begin{aligned}W(t)&=\int_0^t\int_{S_{t'}}\left(\frac{d}{dt}(\Lambda^{-\alpha}\check{\rho})^2+an\mathrm{tr}\chi(\Lambda^{-\alpha}\check{\rho})^2\right)d\mu_\gamma dt'\\&=\int_0^t\int_{S_{t'}}\Big(2[\mathcal{D}_t,\Lambda^{-\alpha}]_g\check{\rho}\cdot\Lambda^{-\alpha}\check{\rho}+(an\mathrm{tr}\chi+\alpha\overline{an\mathrm{tr}\chi})(\Lambda^{-\alpha}\check{\rho})^2\\&\qquad+2\Lambda^{-\alpha}\mathcal{D}_t\check{\rho}\cdot\Lambda^{-\alpha}\check{\rho}\Big)d\mu_\gamma dt'.\end{aligned}\tag{3.8}$$

Let $\vartheta(t)$ be a smooth cut-off function with $\vartheta(0)=1$ and supported in $[0,\frac{1}{2}]$, $|\vartheta|\leq 1$,

$$(\Lambda^{-\alpha}\check{\rho}(0))^2=(\vartheta\Lambda^{-\alpha}\check{\rho})(0)^2-(\vartheta\Lambda^{-\alpha}\check{\rho})(1)^2 \tag{3.9}$$

can be treated similar to $W(t)$, then

$$\begin{aligned} &\|\Lambda^{-\alpha}\check{\rho}\|^2_{L_t^\infty L_x^2} \\ (3.10)\quad &\lesssim \|[\mathcal{D}_t, \Lambda^{-\alpha}]_g\check{\rho}\|_{L_t^1 L_x^2}\cdot\|\Lambda^{-\alpha}\check{\rho}\|_{L_t^\infty L_x^2} + \|(an\mathrm{tr}\chi + \alpha\overline{an\mathrm{tr}\chi})(\Lambda^{-\alpha}\check{\rho})^2\|_{L_t^1 L_x^1} \\ (3.11)\quad &+\int_0^1\left|\int_{S_{t'}}\Lambda^{-\alpha}\mathcal{D}_t\check{\rho}\cdot\Lambda^{-\alpha}\check{\rho}d\mu_\gamma\right|dt' + \int_0^1\left|\int_{S_{t'}}\vartheta^2\Lambda^{-\alpha}\mathcal{D}_t\check{\rho}\cdot\Lambda^{-\alpha}\check{\rho}d\mu_\gamma\right|dt' \\ (3.12)\quad &+\int_0^1\int_{S_{t'}}\left|\vartheta\frac{d}{dt}\vartheta(\Lambda^{-\alpha}\check{\rho})^2\right|d\mu_\gamma dt'. \end{aligned}$$

In view of (2.23) and Proposition 3.1 (ii),

$$\int_0^1\int_{S_{t'}}(|an\mathrm{tr}\chi| + |\overline{an\mathrm{tr}\chi}|)(\Lambda^{-\alpha}\check{\rho})^2 d\mu_{\gamma'}dt' \lesssim \|r^{-\frac12}\Lambda^{-\alpha}\check{\rho}\|^2_{L^2} \lesssim \|\check{\rho}\|^2_{L^2}.$$

By $|\frac{d}{dt}\vartheta| \lesssim 1$ and Proposition 3.1 (ii), for $\alpha \ge \frac12$,

$$(3.12) \lesssim \|\Lambda^{-\alpha}\check{\rho}\|^2_{L^2} \lesssim \|\check{\rho}\|^2_{L^2}.$$

We only need to estimate the first term in (3.11), and the second one will follow similarly.

$$\int_0^1\left|\int_{S_{t'}}\Lambda^{-\alpha}\mathcal{D}_t\check{\rho}\cdot\Lambda^{-\alpha}\check{\rho}d\mu_\gamma\right|dt' \lesssim \|\Lambda^{-2\alpha}\mathcal{D}_t\check{\rho}\|_{L^2}\|\check{\rho}\|_{L^2}. \tag{3.13}$$

Assuming

$$\|\Lambda^{-2\alpha}\mathcal{D}_t\check{\rho}\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{3.14}$$

then (3.13) $\lesssim (\Delta_0^2 + \mathcal{R}_0)^2$, and it follows that

$$\|\Lambda^{-\alpha}\check{\rho}\|^2_{L_t^\infty L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)^2(1 + I_\alpha^{1-\frac{2}{p}})\|\Lambda^{-\alpha}\check{\rho}\|_{L_t^\infty L_x^2} + (\Delta_0^2 + \mathcal{R}_0)^2.$$

Thus (3.6) is proved. (3.4) follows as an immediate consequence.

To prove (3.14), we rely on the transport equation derived by (2.11),

$$\mathcal{D}_t\check{\rho} + \frac32 an\mathrm{tr}\chi\check{\rho} = \mathrm{div}(an\beta) - \nabla\!\!\!/(an)\beta + anA\cdot\tilde{R} = \mathrm{div}(an\beta) + anA\cdot\tilde{R}.$$

By Proposition 3.1, (2.23) and (1.13), with $\alpha \ge \frac12$,

$$\|\Lambda^{-2\alpha}(an\mathrm{tr}\chi\check{\rho})\|_{L^2} \lesssim \|r^{2\alpha}\mathrm{tr}\chi\check{\rho}\|_{L^2} \lesssim \|\check{\rho}\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

By Proposition 3.1 and (1.13), for $\alpha \ge \frac12$,

$$\|\Lambda^{-2\alpha}\mathrm{div}(an\beta)\|_{L^2} \lesssim \|r^{2\alpha-1}an\beta\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{3.15}$$

By Proposition 3.1 (v) and Hölder inequality, we obtain

$$\begin{aligned} \int_0^1\|\Lambda^{-2\alpha}(anA\cdot\tilde{R})\|^2_{L^2(S_{t'})}dt' &= \int_0^1\int_{S_{t'}}\Lambda^{-4\alpha}(anA\cdot\tilde{R})\cdot(anA\cdot\tilde{R})d\mu_\gamma dt' \\ &\le \|\Lambda^{-4\alpha}(anA\cdot\tilde{R})\|_{L_t^2 L_x^4}\|anA\cdot\tilde{R}\|_{L_t^2 L_x^{4/3}} \\ &\lesssim \|\Lambda^{-4\alpha+\frac12+}(anA\cdot\tilde{R})\|_{L_t^2 L_x^2}\|A\|_{L_t^\infty L_x^4}\|\tilde{R}\|_{L^2}. \end{aligned}$$

Consequently, by (**SobM1**), (2.79), (2.53) and Proposition 3.1 (ii),

$$\int_0^t\|\Lambda^{-2\alpha}(anA\cdot\tilde{R})\|^2_{L^2(S_{t'})}dt' \lesssim \|r^{2\alpha-\frac12-}\Lambda^{-2\alpha}(anA\cdot\tilde{R})\|_{L^2(\mathcal{H})}(\Delta_0^2 + \mathcal{R}_0),$$

which implies

$$\|\Lambda^{-2\alpha}(anA\cdot\tilde{R})\|_{L^2(\mathcal{H})}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{3.16}$$

Thus we complete the proof of (3.14). ☐

Proof of Proposition 3.3. By definition (3.1) and Proposition 2.1,

$$\begin{aligned}
&r^{-\alpha}[\Lambda^{-\alpha},\mathcal{D}_t]_g f\\
&=C_\alpha\int_0^\infty d\tau\tau^{\frac{\alpha}{2}-1}e^{-\tau}\int_0^\tau r^2U(\tau-\tau')([\not\Delta,\mathcal{D}_t]U(\tau')f-\overline{an\mathrm{tr}\chi}\not\Delta U(\tau')f)d\tau'\\
&=C_\alpha\int_0^\infty d\tau\tau^{\frac{\alpha}{2}-1}e^{-\tau}\int_0^\tau r^2U(\tau-\tau')(\not\nabla\phi_1(\tau')+\phi_2(\tau')), \qquad (3.17)
\end{aligned}$$

where

$$\begin{aligned}
\phi_1(\tau')&=an(\hat{\chi}+\kappa)\cdot\not\nabla U(\tau')f,\\
\phi_2(\tau')&=\{an(\beta+\not\nabla A+A\cdot A+r^{-1}A)+\not\nabla(an\kappa)\}\not\nabla U(\tau')f.
\end{aligned}$$

Noticing that κ can be regarded as an element of A, thus we have $\not\nabla(an\kappa)=anA\cdot A+an\not\nabla A$, and

$$\phi_1(\tau')=anA\cdot\not\nabla U(\tau')f,\quad \phi_2(\tau')=an(\beta+\not\nabla A+A\cdot A+r^{-1}A)\not\nabla U(\tau')f. \tag{3.18}$$

Let us set

$$\begin{aligned}
\Phi_1&=C_\alpha\int_0^\infty d\tau\tau^{\frac{\alpha}{2}-1}e^{-\tau}\int_0^\tau r^2U(\tau-\tau')\not\nabla\phi_1(\tau')d\tau'\\
\Phi_2&=C_\alpha\int_0^\infty d\tau\tau^{\frac{\alpha}{2}-1}e^{-\tau}\int_0^\tau r^2U(\tau-\tau')\phi_2(\tau')d\tau'.
\end{aligned}$$

The difference between ϕ_1, ϕ_2 in (3.18) and those in [10, Page 41] is the extra factor "an" in (3.18), which can be easily treated by using the estimates $\|\not\nabla(an)\|_{L_t^\infty L_x^4}\lesssim\Delta_0^2+\mathcal{R}_0$ and (2.24). Based on the estimate $\mathcal{N}_1(A)\lesssim\Delta_0^2+\mathcal{R}_0$ which has been proved in (2.79) and Proposition 2.7, also using (2.24), we can proceed exactly as [2, Appendix] or [10, pages 41–43] to obtain for $\frac{1}{2}\le\alpha<1$ and $p>2$,

$$\|\Phi_1\|_{L_t^1L_x^2}+\|\Phi_2\|_{L_t^1L_x^2}\lesssim\|f\|_{L^2}(1+I_\alpha^{1-\frac{2}{p}})(\Delta_0^2+\mathcal{R}_0).$$

The proof is therefore complete. ☐

3.1. L^2 **estimates for Hodge operators.** Consider the following Hodge operators [5]on 2-surfaces $S:=S_t$

- The operator $\mathcal{D}_1$ takes any 1-forms F into the pairs of functions $(\mathrm{div}F,\ \mathrm{curl}\,F)$.
- The operator $\mathcal{D}_2$ takes any symmetric traceless 2-tensors F on S into the 1-forms $\mathrm{div}F$.
- The operator ${}^\star\mathcal{D}_1$ takes the pairs of scalar functions (ρ,σ) into the 1-forms $-\not\nabla\rho+(\not\nabla\sigma)^\star$ on S.
- The operator ${}^\star\mathcal{D}_2$ takes 1-forms F on S into the 2-covariant, symmetric, traceless tensors $-\frac{1}{2}\widehat{\mathcal{L}_F\gamma}$, where

$$(\widehat{\mathcal{L}_F\gamma})_{ab}=\not\nabla_bF_a+\not\nabla_aF_b-(\mathrm{div}F)\gamma_{ab}.$$

Using Proposition 3.2 and Böchner identity, we can follow the the same way as [2, 10] to obtain elliptic estimates for Hodge operators.

[5]For various properties of these operators please refer to [1, Page 38] and [2, Section 4].

Proposition 3.4. *The following estimates hold on $\mathcal{H}$,*

(i) *Let $\mathcal{D}$ denote either $\mathcal{D}_1$ or $\mathcal{D}_2$. The operator $\mathcal{D}$ is invertible on its range, for S tangent tensor F in the range of $\mathcal{D}$,*
$$\|\nabla\!\!\!/ \mathcal{D}^{-1}F\|_{L^2(S)} + \|r^{-1}\mathcal{D}^{-1}F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.$$

(ii) *The operator $(-\Delta\!\!\!/)$ is invertible on its range and its inverse $(-\Delta\!\!\!/)^{-1}$ verifies the estimate*
$$\|\nabla\!\!\!/^2(-\Delta\!\!\!/)^{-1}f\|_{L^2(S)} + \|r^{-1}\nabla\!\!\!/(-\Delta\!\!\!/)^{-1}f\|_{L^2(S)} \lesssim \|f\|_{L^2(S)}.$$

(iii) *The operator ${}^\star\mathcal{D}_1$ is invertible as an operator defined for pairs of H^1 functions with mean zero (i.e. the quotient of H^1 by the kernel of ${}^\star\mathcal{D}_1$) and its inverse ${}^\star\mathcal{D}_1^{-1}$ takes S-tangent L^2 1-forms F(i.e. the full range of ${}^\star\mathcal{D}_1$) into pair of functions (ρ,σ) with mean zero, such that $-\nabla\!\!\!/\rho + (\nabla\!\!\!/\sigma)^\star = F$, verifies the estimate*
$$\|\nabla\!\!\!/ {}^\star\mathcal{D}_1^{-1}F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)},$$
and by (i) and duality argument
$$\|r^{-1\star}\mathcal{D}_1^{-1}F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.$$

(iv) *The operator ${}^\star\mathcal{D}_2$ is invertible as an operator defined on the quotient of H^1-vector fields by the kernel of ${}^\star\mathcal{D}_2$. Its inverse ${}^\star\mathcal{D}_2^{-1}$ takes S-tangent 2-forms Z which is in L^2 space into S tangent 1-forms F (orthogonal to the kernel of $\mathcal{D}_2$), such that ${}^\star\mathcal{D}_2F = Z$, verifies the estimate*
$$\|\nabla\!\!\!/ {}^\star\mathcal{D}_2^{-1}Z\|_{L^2(S)} \lesssim \|Z\|_{L^2(S)}.$$

As a consequence of (i)-(iv), let $\mathcal{D}^{-1}$ be one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, ${}^\star\mathcal{D}_1^{-1}$ or ${}^\star\mathcal{D}_2^{-1}$. By duality argument, we have the following estimate for appropriate[6] *tensor fields F,*
$$\|\mathcal{D}^{-1}divF\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}.$$

Using Proposition 3.2 and Lemma 2.8 and following the similar argument in [2, Proposition 4.24 and Lemma 6.14] we can obtain

Lemma 3.1. *Let $\mathcal{D}$ one of the operators $\mathcal{D}_1$, $\mathcal{D}_2$ and ${}^\star\mathcal{D}_1$. For F pairs of scalar functions in the first case, S-tangent one form for the second and third case, there hold*
$$\mathcal{N}_2(\mathcal{D}^{-1}F) \lesssim \mathcal{N}_1(F) \quad \textit{and} \quad \mathcal{N}_1(\nabla\!\!\!/\mathcal{D}^{-1}F) \lesssim \mathcal{N}_1(F).$$

4. A brief review of theory of geometric Littlewood Paley

Consider $\mathcal{S}$ the collection of smooth functions on $[0,\infty)$ vanishing sufficiently fast at ∞ and verifying the vanishing moment property
$$\int_0^\infty \tau^{k_1}\partial^{k_2}m(\tau)d\tau = 0, \quad k_1 + k_2 \le N.$$
We set $m_k(\tau) := 2^{2k}m(2^{2k}\tau)$ for some smooth function $m \in \mathcal{S}$. Recall from [3] the geometric Littlewood-Paley (GLP) projections P_k associated to m which take the form
$$P_kF := \int_0^\infty m_k(\tau)U(\tau)Fd\tau$$

[6] By " appropriate" , we mean the tensor F such that div$F$ is in the space where $\mathcal{D}^{-1}$ is well-defined.

for any S_t tangent tensor field F, where $U(\tau)F$ is defined by the heat flow (3.2) on $(S_t, \overset{\circ}{\gamma})$.

Proposition 4.1. [7] *There exists $m \in \mathcal{S}$ such that the GLP projections P_k associated to m verify $U(\infty) + \sum_{k\in\mathbb{Z}} P_k^2 = Id$. By f we denote a scalar function and F a S-tangent tensor field on $\mathcal{H}$, the GLP projections P_k associated to arbitrary induced function m verify the following properties:*

(i) *(L^p-boundedness) For any $1 \le p \le \infty$, and any interval $I \subset \mathbb{Z}$,*

$$\|P_I F\|_{L^p(S)} \lesssim \|F\|_{L^p(S)}$$

(ii) *(Bessel inequality) For any tensorfield F on S,*

$$\sum_k \|P_k F\|^2_{L^2(S)} \lesssim \|F\|^2_{L^2(S)}, \quad \sum_k 2^{2k} r^{-2} \|P_k F\|^2_{L^2(S)} \lesssim \|\nabla\!\!\!\!/ F\|^2_{L^2(S)}$$

(iii) *(Finite band property) For any $1 \le p \le \infty$, $k \ge 0$,*

$$\|\Delta\!\!\!\!/ P_k F\|_{L^p(S)} \lesssim 2^{2k} r^{-2} \|F\|_{L^p(S)}. \tag{4.1}$$

Moreover given $m \in \mathcal{S}$ we can find $\tilde{m} \in \mathcal{S}$ such that $2^{2k} P_k = \Delta\!\!\!\!/ \tilde{P}_k$, with $\tilde{P}_k$ the geometric Littlewood Paley projections associated to $\tilde{m}$, then

$$P_k F = 2^{-2k} \tilde{P}_k \Delta\!\!\!\!/ F, \quad \|P_k F\|_{L^p(S)} \lesssim 2^{-2k} r^2 \|\Delta\!\!\!\!/ F\|_{L^p(S)}. \tag{4.2}$$

In addition, there hold L^2 estimates

$$\begin{cases} \|\nabla\!\!\!\!/ P_k F\|_{L^2(S)} \lesssim 2^k r^{-1} \|F\|_{L^2(S)} \\ \|P_k \nabla\!\!\!\!/ F\|_{L^2(S)} \lesssim 2^k r^{-1} \|F\|_{L^2(S)} \\ \|\nabla\!\!\!\!/ P_{\le 0} F\|_{L^2(S)} \lesssim r^{-1} \|F\|_{L^2(S)}, \end{cases} \tag{4.3}$$

and

$$\|P_k F\|_{L^2(S)} \lesssim 2^{-k} r \|\nabla\!\!\!\!/ F\|_{L^2(S)} \tag{4.4}$$

(iv) *(Bernstein Inequality) For appropriate tensor fields F on S, we have weak Bernstein inequalities*

$$\begin{cases} \|P_k F\|_{L^p(S)} \lesssim r^{\frac{2}{p}-1} \left(2^{(1-\frac{2}{p})k} + 1\right) \|F\|_{L^2(S)}, \\ \|P_{\le 0} F\|_{L^p(S)} \lesssim r^{\frac{2}{p}-1} \|F\|_{L^2(S)} \\ \|P_k F\|_{L^2(S)} \lesssim r^{\frac{2}{p}-1} \left(2^{(1-\frac{2}{p})k} + 1\right) \|F\|_{L^{p'}(S)} \\ \|P_{\le 0} F\|_{L^2(S)} \lesssim r^{\frac{2}{p}-1} \|F\|_{L^{p'}(S)}. \end{cases} \tag{4.5}$$

where $2 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $k \ge 0$.

(v) *With the help of Proposition 3.2, we have the sharp Bernstein inequalities*

$$\|P_k f\|_{L^\infty(S)} \lesssim 2^k r^{-1} \|f\|_{L^2(S)}, \quad \|P_k f\|_{L^2(S)} \lesssim 2^k r^{-1} \|f\|_{L^1(S)}. \tag{4.6}$$

We will also use the notations for any S-tangent tensor field F,

$$F_n := P_n^2 F, \quad F_{\le 0} := \sum_{k \le 0} P_k^2 F. \tag{4.7}$$

[7] For more properties, one can refer to [3, 4] and [10].

Now we define for $0 \le \theta \le 1$ the Besov $\mathcal{B}^\theta$, $\mathcal{P}^\theta$ norms for S-tangent tensor fields F on $\mathcal{H}$ and $B^\theta_{2,1}$ norm on S as follows:

$$\|F\|_{\mathcal{B}^\theta} = \sum_{k>0} \|(2^k r^{-1})^\theta P_k F\|_{L_t^\infty L_x^2} + \|r^{-\theta} F\|_{L_t^\infty L_x^2}, \tag{4.8}$$

$$\|F\|_{\mathcal{P}^\theta} = \sum_{k>0} \|(2^k r^{-1})^\theta P_k F\|_{L_t^2 L_x^2} + \|r^{-\theta} F\|_{L_t^2 L_x^2}, \tag{4.9}$$

$$\|F\|_{B^\theta_{2,1}(S)} = \sum_{k>0} \|(2^k r^{-1})^\theta P_k F\|_{L_x^2} + \|r^{-\theta} F\|_{L_x^2}. \tag{4.10}$$

Remark 4.1. For any $a \in \mathbb{R}$ and any S-tangent tensor field F

$$\|F\|^2_{H^a(S)} := \|\Lambda^a F\|^2_{L^2(S)} \approx \sum_{k>0} 2^{2ka} r^{-2a} \|P_k F\|^2_{L^2(S)} + r^{-2a} \|P_{\le 0} F\|^2_{L_x^2}.$$

In certain situation, it is more convenient to work with the Besov norms defined by the classical Littlewood-Paley (LP) projections E_k. Recall that (see [8, 9]) for any scalar function f on $\mathbb{R}^2$ we can define

$$E_k f = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \varphi(\xi/2^k) \hat{f}(\xi) e^{ix\xi} d\xi, [8]$$

where φ is a smooth function support in the dyadic shell $\{\frac{1}{2} \le |\xi| \le 2\}$ and satisfying $\sum_{k\in\mathbb{Z}} \varphi(2^{-k}\xi) = 1$ when $\xi \neq 0$.

Define for any $0 \le \theta < 1$ the $\tilde{\mathcal{B}}^\theta$ and $\tilde{\mathcal{P}}^\theta$ norms of any scalar function f on $\mathcal{H}$ by

$$\|f\|_{\tilde{\mathcal{B}}^\theta} := \sum_{k>0} \|(2^k r^{-1})^\theta E_k f\|_{L_t^\infty L_x^2} + \|r^{-\theta} f\|_{L_t^\infty L_x^2}, \tag{4.11}$$

$$\|f\|_{\tilde{\mathcal{P}}^\theta} := \sum_{k>0} \|(2^k r^{-1})^\theta E_k f\|_{L_t^2 L_x^2} + \|r^{-\theta} f\|_{L_t^2 L_x^2}. \tag{4.12}$$

Using Proposition 2.8 and BA1 we can adapt [7, Proposition 3.28] to obtain the following lemma.

Lemma 4.1. *Under the bootstrap assumptions (BA1) , there exists a finite number of vector fields $\{X_i\}_{i=1}^l$ verifying the conditions*

$$\begin{cases} \|X, r\nabla_0 X\|_{L_t^\infty L_\omega^\infty} \lesssim 1, \quad \|r\nabla\!\!\!\!/(\nabla_0 X)\|_{L_x^2 L_t^\infty} \lesssim 1, \\ \|(\nabla\!\!\!\!/ - \nabla_0) X\|_{L_x^2 L_t^\infty} \lesssim \Delta_0, \quad \nabla\!\!\!\!/_L X = 0, \end{cases}$$

where ∇_0 represents the covariant derivative induced by the metric $r^2\gamma^{(0)}$. For appropriate S-tangent tensor $F \in L_t^\infty L_x^2$, $F \in \mathcal{B}^\theta$ if and only if $F \cdot X_i \in \mathcal{B}^\theta$, and

$$C^{-1} \sum_i \|F \cdot X_i\|_{\mathcal{B}^\theta} \le \|F\|_{\mathcal{B}^\theta} \le C \sum_i \|F \cdot X_i\|_{\mathcal{B}^\theta}, \text{ with } 0 \le \theta < 1,$$

where C is a positive constant. The same results hold for the spaces $\mathcal{P}^\theta$. Moreover

$$\mathcal{N}_1(F \otimes X) + \|F \otimes X\|_{L_\omega^\infty L_t^2} \lesssim \mathcal{N}_1(F) + \|F\|_{L_\omega^\infty L_t^2},$$

where $\otimes$ stands for either a tensor product or a contraction.

Lemma 4.1 allows us to define Besov norms for arbitrary S-tangent tensor fields F on $\mathcal{H}$ by the classical LP projections.

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[8]See [3, Page 2] for the finite band and sharp Berstein inequalities of E_k.

Definition 4.2. Let F be an (m,n) S-tangent tensor field on $\mathcal{H}$ and let $F^{j_1j_2\cdots j_m}_{i_1i_2\cdots i_n}$ be the local components of F relative to $\{X_i\}_{i=1}^l$. We define the $\tilde{\mathcal{B}}^\theta$ and $\tilde{\mathcal{P}}^\theta$ norms of F by

$$\|F\|_{\tilde{\mathcal{B}}^\theta} = \sum \|F^{j_1j_2\cdots j_m}_{i_1i_2\cdots i_n}\|_{\tilde{\mathcal{B}}^\theta} \quad \text{and} \quad \|F\|_{\tilde{\mathcal{P}}^\theta} = \sum \|F^{j_1j_2\cdots j_m}_{i_1i_2\cdots i_n}\|_{\tilde{\mathcal{P}}^\theta},$$

where the summation is taken over all possible $(i_1\cdots i_n; j_1\cdots j_m)$.

The equivalence between $\mathcal{B}^\theta$, $\mathcal{P}^\theta$ norms and $\tilde{\mathcal{B}}^\theta$, $\tilde{\mathcal{P}}^\theta$ norms is given in the following result whose proof can be found in [10].

Proposition 4.2. *Under the bootstrap assumptions BA1 for arbitrary S-tangent tensor fields F on $\mathcal{H}$ there hold for $0 \le \theta < 1$,*

$$\|F\|_{\tilde{\mathcal{B}}^\theta} \approx \|F\|_{\mathcal{B}^\theta} \quad \textit{and} \quad \|F\|_{\tilde{\mathcal{P}}^\theta} \approx \|F\|_{\mathcal{P}^\theta}.$$

Lemma 4.2. *Let H be any S_t tangent tensor and let f be a smooth function,*

$$\|fH\|_{\mathcal{P}^0} \lesssim \|H\|_{\mathcal{P}^0}\|f\|_{L^\infty} + \|H\|_{L^2(\mathcal{H})}\|r^{1/2}\nabla\!\!\!/ f\|_{L_t^\infty L_x^4}. \tag{4.13}$$

and the similar estimates hold for $\mathcal{B}^0$ and $B^0_{2,1}(S)$.

Proof. By GLP decomposition, $H = \bar{H} + \sum_{n\in\mathbb{N}} P_n^2 H + P_{\le 0}H$, where $\bar{H} = U(\infty)H$.

(4.14)

$$\|fH\|_{\mathcal{P}^0} \le \sum_{k>0}\|P_k(fH_{<k})\|_{L^2} + \sum_{k>0}\|P_k(fH_{\ge k})\|_{L^2} + \sum_{k>0}\|P_k(f\bar{H})\|_{L^2} + \|fH\|_{L^2}.$$

Let $H_n := P_n^2 H$. Consider the first term in (4.14). By (4.4), (4.5) and (4.3),

$$\begin{aligned}
\sum_{k>0,k>n>0} \|P_k(fH_n)\|_{L^2} &\lesssim \sum_{k>0,k>n>0} 2^{-k}\|rP_k\nabla\!\!\!/(fH_n)\|_{L^2} \\
&\lesssim \sum_{k>0,k>n>0} 2^{-k}\left(\|rP_k(\nabla\!\!\!/ f\cdot H_n)\|_{L^2} + \|rP_k(f\cdot\nabla\!\!\!/ H_n)\|_{L^2}\right) \\
&\lesssim \sum_{k>0,k>n>0}\left(2^{-k+\frac{k}{2}}\|r^{\frac12}\nabla\!\!\!/ f\cdot H_n\|_{L_t^2L_x^{4/3}} + 2^{-k+n}\|f\|_{L^\infty}\|H_n\|_{L^2}\right) \\
&\lesssim \sum_{k>0,k>n>0}\left(2^{-\frac{k}{2}}\|r^{\frac12}\nabla\!\!\!/ f\|_{L_t^\infty L_x^4}\|H_n\|_{L^2} + 2^{-k+n}\|f\|_{L^\infty}\|P_nH\|_{L^2}\right) \\
&\lesssim \|H\|_{L^2}\|r^{1/2}\nabla\!\!\!/ f\|_{L_t^\infty L_x^4} + \|f\|_{L^\infty}\|H\|_{\mathcal{P}^0},
\end{aligned}$$

and similarly,

$$\sum_{k>0}\|P_k(fH_{\le 0})\|_{L^2} \lesssim \|H\|_{L^2}\|r^{1/2}\nabla\!\!\!/ f\|_{L_t^\infty L_x^4} + \|f\|_{L^\infty}\|H\|_{\mathcal{P}^0}.$$

Now consider the second term. By (4.2), (4.3) and (4.5), we obtain

$$\begin{aligned}
&\sum_{k>0,n\ge k}\|P_k(fH_n)\|_{L^2} \\
&\quad\lesssim \sum_{k>0,n\ge k} 2^{-2n}\|r^2P_k(f\Delta\!\!\!/ H_n)\|_{L^2} \\
&\quad\lesssim \sum_{k>0,n\ge k} 2^{-2n}\left(\|r^2P_k\nabla\!\!\!/(f\nabla\!\!\!/ H_n)\|_{L^2} + \|r^2P_k(\nabla\!\!\!/ f\cdot\nabla\!\!\!/ H_n)\|_{L^2}\right) \\
&\quad\lesssim \sum_{k>0,n\ge k}\left(2^{-2n+k+n}\|f\|_{L^\infty}\|P_nH\|_{L^2} + 2^{-2n+\frac{k}{2}}\|r^{\frac32}\nabla\!\!\!/ f\cdot\nabla\!\!\!/ H_n\|_{L_t^2L_x^{4/3}}\right).
\end{aligned}$$

By (4.3), we have

$$2^{-2n+k/2}\|r^{\frac{3}{2}}\nabla\!\!\!/ f\cdot\nabla\!\!\!/ H_n\|_{L_t^2L_x^{4/3}}\lesssim 2^{-2n+\frac{k}{2}}\|r^{1/2}\nabla\!\!\!/ f\|_{L_t^\infty L_x^4}\|r\nabla\!\!\!/ H_n\|_{L^2}$$
$$\lesssim 2^{\frac{k}{2}-n}\|r^{1/2}\nabla\!\!\!/ f\|_{L_t^\infty L_x^4}\|H\|_{L^2},$$

thus

$$\sum_{k>0,n\ge k}\|P_k(fH_n)\|_{L^2}\lesssim\|H\|_{L^2}\|r^{\frac{1}{2}}\nabla\!\!\!/ f\|_{L_t^\infty L_x^4}+\|f\|_{L^\infty}\|H\|_{\mathcal{P}^0}.$$

At last, due to $\nabla\!\!\!/ \bar{H}=0$ and (**SobM1**), we can deduce

$$\sum_{k>0}\|P_k(f\bar{H})\|_{L^2}\lesssim\sum_{k>0}2^{-k}\|r^{\frac{1}{2}}\nabla\!\!\!/ f\|_{L_t^\infty L_x^4}\|\bar{H}\|_{L^2}\lesssim\|r^{\frac{1}{2}}\nabla\!\!\!/ f\|_{L_t^\infty L_x^4}\|H\|_{L^2}.$$

The proof is complete. □

4.1. **Product estimates and Intertwine estimates.**

Proposition 4.3. *Let $\mathcal{D}$ be one of the operators $\mathcal{D}_1$, $\mathcal{D}_2$ and ${}^\star\mathcal{D}_1$. Then for $1<p\le 2$ and any S-tangent tensor F on $\mathcal{H}$ there holds*

$$\|\mathcal{D}^{-1}F\|_{L^2(S)}\lesssim\|r^{2-\frac{2}{p}}F\|_{L^p(S)}.$$

Proof. For $p=2$, the inequality follows immediately from Proposition 3.4. For $p>2$, from (**Sob**) and Proposition 3.4 we infer for $p'>2$ satisfying $\frac{1}{p}+\frac{1}{p'}=1$ that

$$\|r^{\frac{2}{p}-2\star}\mathcal{D}^{-1}F\|_{L^{p'}(S)}\lesssim\|\nabla\!\!\!/^\star\mathcal{D}^{-1}F\|_{L^2(S)}^{1-\frac{2}{p'}}\|r^{-1\star}\mathcal{D}^{-1}F\|_{L^2(S)}^{\frac{2}{p'}}+\|r^{-1\star}\mathcal{D}^{-1}F\|_{L^2(S)}$$
$$\lesssim\|F\|_{L^2(S)}$$

Thus, by duality, we complete the proof. □

Lemma 4.3. *Let $\mathcal{D}$ denote one of the Hodge operators $\mathcal{D}_1$, $\mathcal{D}_2$, ${}^\star\mathcal{D}_1$ and ${}^\star\mathcal{D}_2$, let $\mathcal{D}^{-1}$ denote the inverse of $\mathcal{D}$. For P_kF with P_k the GLP projections associated to the heat equation (3.2), there hold for $k>0$, $1<p\le 2$,*

$$\|\mathcal{D}^{-1}P_kF\|_{L_x^2}\lesssim 2^{-k}r\|F\|_{L_x^2}\quad\text{and}\quad\|P_k\mathcal{D}^{-1}F\|_{L_x^2}\lesssim 2^{-(2-\frac{2}{p})k}r^{2-\frac{2}{p}}\|F\|_{L_x^p}.$$

Proof. The first inequality can be proved by using (4.4) in Proposition 4.1 and Proposition 3.4. The second can be proved by duality with the help of the first inequality and (**SobM1**). □

The following result follows from the second estimate in Lemma 4.3 immediately.

Proposition 4.4. *Let $\mathcal{D}^{-1}$ denote either $\mathcal{D}_1^{-1}$, ${}^\star\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, then for appropriate S-tangent tensor fields F on $\mathcal{H}$ and any $1<p\le 2$,*

$$\|\mathcal{D}^{-1}F\|_{\mathcal{P}^\theta}\lesssim\|r^{2-\frac{2}{p}-\theta}F\|_{L_t^2L_x^p}. \tag{4.15}$$

Since the proof of the Hodge-elliptic estimate for geodesic foliation contained in [11, pages 295–301] only relied on

$$\|\underline{K}\|_{L^2}+\|\Lambda^{-\alpha_0}\underline{K}\|_{L_t^\infty L_x^2}\lesssim\Delta_0^2+\mathcal{R}_0,\ \text{with}\ \alpha_0\ge 1/2.$$

Hence, based on Lemma 2.8 and Proposition 3.2, the same proof also applies to the case of time foliation. We can obtain the result on the Hodge-elliptic $\mathcal{P}^\sigma$ estimates.

Theorem 4.3 (Hodge-elliptic $\mathcal{P}^\sigma$-estimate)**.** *Let $\mathcal{D}$ denote either $\mathcal{D}_1$, $\mathcal{D}_2$ or their adjoint operators ${}^\star\mathcal{D}_1$ and ${}^\star\mathcal{D}_2$. Then for any S-tangent tensor fields ξ and F satisfying $\mathcal{D}\xi = F$ and any $\frac{1}{2} > \sigma \geq 0$,*

$$\|\nabla\!\!\!/\xi\|_{\mathcal{P}^\sigma} \lesssim \|F\|_{\mathcal{P}^\sigma} + \Delta_0 \|\mathcal{D}^{-1}F\|^q_{L_t^b L_x^2} \|F\|^{1-q}_{L_t^2 L_x^2}, \tag{4.16}$$

where $1/2 \leq \alpha_0 < q < 1-\sigma$ and $b > 4$.

We now give a series of product estimates for Besov norms. Proofs can be seen in [11, pages 302–304].

Lemma 4.4. *For any S-tangent tensor fields F and G,*

$$\|F \cdot G\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F)(\|r^{-\frac{1}{b}} G\|_{L_t^b L_x^2} + \|r^{\frac{1}{2}} \nabla\!\!\!/ G\|_{L_t^2 L_x^2}) \text{ with } b > 4, \tag{4.17}$$

$$\|F \cdot G\|_{\mathcal{P}^0} \lesssim \mathcal{N}_2(r^{1/2} F)\|G\|_{\mathcal{P}^0}, \tag{4.18}$$

$$\|F \cdot G\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(r^{\frac{1}{2}} F)\big(\|\nabla\!\!\!/ G\|_{L_t^2 L_x^2} + \|G\|_{L_\omega^\infty L_t^2}\big). \tag{4.19}$$

Corollary 1. *Regard κ, ι also as elements of A, there hold*

$$\|A \cdot F\|_{\mathcal{P}^0} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(r^{\frac{1}{2}} F), \qquad \|(tr\chi, r^{-1})F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F), \tag{4.20}$$

$$\|anA \cdot F\|_{\mathcal{P}^0} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(r^{\frac{1}{2}} F), \quad \|an(tr\chi, r^{-1})F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F). \tag{4.21}$$

Proof. Let us prove (4.20) first. Using (4.17) for $b > 4$, we have

$$\|A \cdot F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(A)(\|r^{\frac{1}{2}} \nabla\!\!\!/ F\|_{L^2(H)} + \|r^{-\frac{1}{b}} F\|_{L_t^b L_x^2}) \lesssim \mathcal{N}_1(A) \cdot \mathcal{N}_1(r^{\frac{1}{2}} F) \tag{4.22}$$

where the last inequality follows by using (2.28).

By finite band property of GLP in Proposition 4.1, it is straightforward to obtain

$$\|r^{-1} F\|_{\mathcal{P}^0} \lesssim \|\nabla\!\!\!/ F\|_{L^2} + \|r^{-1} F\|_{L^2}. \tag{4.23}$$

Noticing that $\mathrm{tr}\chi \cdot F = \iota \cdot F + 2r^{-1} \cdot F$ and $\mathcal{N}_1(\iota) \lesssim \Delta_0^2 + \mathcal{R}_0$ in Proposition 2.7, the other inequality in (4.20) follows by (4.22) and (4.23) with A replaced by ι.

Applying (4.13) to $f = an$ and $H = A \cdot F$, $\mathrm{tr}\chi F$, $r^{-1}F$, (4.21) can be derived by using (4.20) for H and the following inequalities for f

$$\|\nabla\!\!\!/(an)\|_{L_t^\infty L_x^4} \approx \|\zeta + \underline{\zeta}\|_{L_t^\infty L_x^4} \lesssim \Delta_0^2 + \mathcal{R}_0, \quad \|an\|_{L^\infty} \lesssim 1. \tag{4.24}$$

□

5. **Sharp trace theorem**

The purpose of the section is to prove

Theorem 5.1 (Sharp trace theorem)**.** *Let F be an S-tangent tensor which admits a decomposition of the form $\nabla\!\!\!/(anF) = \mathcal{D}_t P + E$ with tensors P and E of the same type as F, suppose $\lim_{t\to 0} \|F\|_{L_x^\infty} < \infty$ and $\lim_{t\to 0} r|\nabla\!\!\!/ F| < \infty$, there holds the following sharp trace inequality,*

$$\|F\|_{L_\omega^\infty L_t^2} \lesssim \mathcal{N}_1(F) + \mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0}. \tag{5.1}$$

Remark 5.2. We will employ this theorem to estimate $\|F\|_{L_\omega^\infty L_t^2}$ for $F = \underline{\zeta}, \nu, \hat{\chi}, \zeta$. By local analysis, the two initial assumptions can be checked for the four quantities.

The following result gives the important inequalities to prove Theorem 5.1.

Proposition 5.1. *Let $p \ge 1$ be any integer, for any S-tangent tensor fields F, H and G of the same type. There holds*

$$\left\| r^{-p} \int_0^t r'^p H \cdot G dt' \right\|_{\mathcal{B}^0} \lesssim \|H\|_{\mathcal{P}^0} (\mathcal{N}_1(G) + \|G\|_{L_\omega^\infty L_t^2}). \tag{5.2}$$

Let us further assume

$$\lim_{t\to 0} r^p \|F\|_{L_x^\infty} = 0, \quad \lim_{t\to 0} \|G\|_{L_x^\infty} < \infty, \text{ when } p \ge 2, \tag{5.3}$$

then there holds for $p \ge 1$,

$$\left\| r^{-p} \int_0^t r'^p \mathcal{D}_t F \cdot G dt' \right\|_{\mathcal{B}^0} \lesssim \mathcal{N}_1(F)\mathcal{N}_1(G). \tag{5.4}$$

Using a modified version of [4, Lemma 5.3], (see in Lemma 8.1 in Appendix), also using Proposition 4.2, Proposition 5.1 follows by repeating the procedure in [4] and [10, Appendix]. We omit the detail of the proof of Proposition 5.1.

Proof of Theorem 5.1. We set $\varphi(t) = \int_0^t an|F|^2 dt'$, then $\mathcal{D}_t \varphi = an|F|^2$. Due to [2, Proposition 5.1] we have

$$\|\varphi\|_{L^\infty(\mathcal{H})} \lesssim \|\nabla\!\!\!/ \varphi\|_{\mathcal{B}^0} + \|r^{-1}\varphi\|_{L_t^\infty L_x^2}. \tag{5.5}$$

It is easy to see

$$\|r^{-1}\varphi\|_{L_t^\infty L_x^2} \lesssim \|F\|_{L_\omega^\infty L_t^2} \cdot \|r^{-1} F\|_{L^2}. \tag{5.6}$$

We now estimate $\|\nabla\!\!\!/ \varphi\|_{\mathcal{B}^0}$. In view of (2.19), we obtain

$$\nabla\!\!\!/_L \nabla\!\!\!/ \varphi + \frac{1}{2}\mathrm{tr}\chi \nabla\!\!\!/ \varphi = 2(an)^{-1} \nabla\!\!\!/ (an \cdot F) \cdot F - \hat{\chi} \cdot \nabla\!\!\!/ \varphi - (\zeta + \underline{\zeta})|F|^2. \tag{5.7}$$

Using the decomposition $\nabla\!\!\!/ (an \cdot F) = \mathcal{D}_t P + E$ and Lemma 4.1, we pair $\nabla\!\!\!/ \varphi$ with vector field $X = X_i$ to obtain

$$\begin{aligned} \nabla\!\!\!/_L (r \nabla\!\!\!/ \varphi \cdot X) &= -\frac{1}{2} r\kappa \nabla\!\!\!/ \varphi \cdot X - r\hat{\chi} \nabla\!\!\!/ \varphi \otimes X \\ &\quad + r\left(2\nabla\!\!\!/_L P \cdot F \otimes X + 2(an)^{-1} E \cdot F \otimes X - |F|^2 (\zeta + \underline{\zeta}) \cdot X\right). \end{aligned} \tag{5.8}$$

We will not distinguish "$\otimes$" with "$\cdot$", and also suppress X whenever there occurs no confusion. Note that under the transport coordinate (s, ω_1, ω_2), we have

$$\frac{\partial \varphi}{\partial \omega_i} = \int_0^t \{2\langle \partial_{\omega_i} F, F\rangle + |F|^2 \partial_{\omega_i} \log(an)\} na dt' \quad i = 1, 2,$$

by assumptions on initial condition, $\lim_{t\to 0} \frac{\partial \varphi}{\partial \omega_i} = 0$. It then follows by integrating (5.8) along a null geodesic Γ_ω from 0 to $s(t)$ that, symbolically,

$$\begin{aligned} (\nabla\!\!\!/ \varphi)(t) &= r^{-1} \int_0^t \{r' an(\kappa + \hat{\chi}) \nabla\!\!\!/ \varphi + r' E \cdot F\} dt' \\ &\quad + r^{-1} \int_0^t r' \mathcal{D}_t P \cdot F dt' + r^{-1} \int_0^t an r' |F|^2 (\zeta + \underline{\zeta}) dt'. \end{aligned} \tag{5.9}$$

Since by Proposition 4.2,

$$\|\nabla\!\!\!/ \varphi\|_{\mathcal{B}^0} \approx \sum_{k>0} \|E_k(\nabla\!\!\!/ \varphi)\|_{L_t^\infty L_x^2} + \|\nabla\!\!\!/ \varphi\|_{L_t^\infty L_x^2}, \tag{5.10}$$

and the estimate of $\|\nabla\!\!\!/ \varphi\|_{L_x^2 L_t^\infty}$ can be obtained in view of (5.9) and (**SobM1**).

$$\|\nabla\!\!\!/ \varphi\|_{L_x^2 L_t^\infty} \lesssim (\|\kappa\|_{L_\omega^\infty L_t^2} + \|\hat\chi\|_{L_\omega^\infty L_t^2})\|\nabla\!\!\!/ \varphi\|_{L^2} + (\|E\|_{L^2} + \|\nabla\!\!\!/_L P\|_{L^2} + \mathcal{N}_1(A)\mathcal{N}_1(F))\|r^{-1}F\|_{L^2} \tag{5.11}$$

where the term on the right of (5.11) can be absorbed in view of $\|\kappa, \hat\chi\|_{L_\omega^\infty L_t^2} \lesssim \Delta_0$, obtained from (2.84) and BA1. It remains to estimate the first term in (5.10).

Applying Littlewood Paley decomposition E_k to $\nabla\!\!\!/ \varphi \otimes X$ via (5.9), we only need to estimate the following terms

$$I_1 = \sum_{k>0}\left\|E_k\int_0^t r' an(\kappa+\hat\chi)\nabla\!\!\!/ \varphi\right\|_{L_t^\infty L_\omega^2}, \quad I_2 = \sum_{k>0}\left\|E_k\int_0^t r' E\cdot F dt'\right\|_{L_t^\infty L_\omega^2},$$
$$I_3 = \sum_{k>0}\left\|E_k\int_0^t r'\mathcal{D}_t P\cdot F dt'\right\|_{L_t^\infty L_\omega^2}, \qquad I_4 = \sum_{k>0}\left\|E_k\int_0^t r'|F|^2(\zeta+\underline{\zeta})dt'\right\|_{L_t^\infty L_\omega^2}.$$

Use (5.2) and Proposition 4.2,

$$I_1 \lesssim (\mathcal{N}_1(\hat\chi) + \|\hat\chi\|_{L_x^\infty L_t^2} + \mathcal{N}_1(\kappa) + \|\kappa\|_{L_\omega^\infty L_t^2})\|\nabla\!\!\!/ \varphi\|_{\mathcal{P}^0}.$$

Consequently, by (2.84), Proposition 2.7, (2.79) and $\|\hat\chi\|_{L_\omega^\infty L_t^2} \le \Delta_0$ in BA1,

$$I_1 \lesssim \Delta_0\|\nabla\!\!\!/ \varphi\|_{\mathcal{P}^0}.$$

Apply (5.2) to I_2,

$$I_2 \lesssim \|E\|_{\mathcal{P}^0}(\mathcal{N}_1(F) + \|F\|_{L_\omega^\infty L_t^2}).$$

By (5.4) and Lemma 4.1

$$I_3 \lesssim \mathcal{N}_1(P)(\mathcal{N}_1(F) + \|F\|_{L_\omega^\infty L_t^2}).$$

By (5.2), (4.19), BA1 and (2.79), we have

$$\begin{aligned} I_4 &\lesssim (\mathcal{N}_1(\zeta+\underline{\zeta}) + \|\zeta+\underline{\zeta}\|_{L_\omega^\infty L_t^2})\|F\cdot F\|_{\mathcal{P}^0} \\ &\lesssim \Delta_0\mathcal{N}_1(F)(\|\nabla\!\!\!/ F\|_{L^2} + \|F\|_{L_\omega^\infty L_t^2}). \end{aligned}$$

Thus we conclude

$$\begin{aligned} \|\nabla\!\!\!/ \varphi\|_{\mathcal{B}^0} \lesssim{}& (\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0})\left(\mathcal{N}_1(F) + \|F\|_{L_\omega^\infty L_t^2}\right) + \Delta_0\|\nabla\!\!\!/ \varphi\|_{\mathcal{P}^0} \\ &+ \Delta_0\mathcal{N}_1(F)(\|\nabla\!\!\!/ F\|_{L^2} + \|F\|_{L_\omega^\infty L_t^2}). \end{aligned}$$

This inequality, together with the fact that $\|\nabla\!\!\!/ \varphi\|_{\mathcal{P}^0} \lesssim \|\nabla\!\!\!/ \varphi\|_{\mathcal{B}^0}$, yields

$$\|\nabla\!\!\!/ \varphi\|_{\mathcal{B}^0} \lesssim (\mathcal{N}_1(F) + \|F\|_{L_\omega^\infty L_t^2})\left(\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} + \Delta_0\mathcal{N}_1(F)\right).$$

Combine the above inequality with (5.5) and (5.6), we get

$$\begin{aligned} \|F\|^2_{L_\omega^\infty L_t^2} \lesssim{}& (\mathcal{N}_1(F) + \|F\|_{L_\omega^\infty L_t^2})\left(\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} + \mathcal{N}_1(F)\Delta_0\right) \\ &+ \|F\|_{L_\omega^\infty L_t^2}\cdot\|r^{-1}F\|_{L^2} \end{aligned}$$

which implies (5.1) by Young's inequality. $\square$

6. **Error estimates**

We will employ the following conventions:

- $\check{R}$ denotes either the pair $(\check{\rho}, -\check{\sigma})$ or $\underline{\beta}$
- $\mathcal{D}^{-1}\check{R}$ denotes either $\mathcal{D}_1^{-1}(\check{\rho}, -\check{\sigma})$ or ${}^\star\mathcal{D}_1^{-1}\underline{\beta}$
- $\mathcal{D}^{-2}\check{R}$ denotes either $\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}(\check{\rho}, -\check{\sigma})$ or $\mathcal{D}_1^{-1}{}^\star\mathcal{D}_1^{-1}\underline{\beta}$
- $\mathcal{D}^{-1}\mathcal{D}_t\check{R}$ denotes either ${}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta}$ or $\mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma})$
- $C_0(\check{R})$ denotes $[\mathcal{D}_t, \mathcal{D}_1^{-1}](\check{\rho}, -\check{\sigma})$ or $[\mathcal{D}_t, {}^\star\mathcal{D}_1^{-1}]\underline{\beta}$
- $\mathcal{D}^{-2}\mathcal{D}_t\check{R}$ denotes $\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma})$ or $\mathcal{D}_1^{-1}{}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta}$
- $\mathcal{D}^{-1}C_0(\check{R})$ denotes $\mathcal{D}_2^{-1}[\mathcal{D}_t, \mathcal{D}_1^{-1}](\check{\rho}, -\check{\sigma})$ or $\mathcal{D}_1^{-1}[\mathcal{D}_t, {}^\star\mathcal{D}_1^{-1}]\underline{\beta}$
- $\mathcal{F}$ denotes $\mathcal{D}^{-1}\check{R}$ or $(a\delta + 2a\lambda)$. [9]
- $\mathcal{D}^{-1}\mathcal{F}$ denotes either $\mathcal{D}^{-2}\check{R}$ or $\mathcal{D}_1^{-1}(a\delta + 2a\lambda)$.

6.1. **Commutation formula.** We will study error terms which arise from commuting $\mathcal{D}_t$ with Hodge operators. Regard ι also an element of A, symbolically, the commutation formula and its good part can be written as follows

$$[\mathcal{D}_t, \nabla\!\!\!/]F = an\left((A + \frac{1}{r})\nabla\!\!\!/ F + (A + \frac{1}{r}) \cdot A \cdot F + \beta \cdot F\right), \tag{6.1}$$

$$[\mathcal{D}_t, \nabla\!\!\!/]_g F := an\left((A + \frac{1}{r})\nabla\!\!\!/ F + (A + \frac{1}{r}) \cdot A \cdot F\right). \tag{6.2}$$

Due to the nontrivial factor "an" in (6.1), the treatment in [11, Section 6] has to be modified. We rewrite equations (2.11), (2.12) and (2.13) as

$$L(\check{\rho}, -\check{\sigma}) = \mathcal{D}_1\beta + r^{-1}\check{R} + A \cdot \tilde{R}, \tag{6.3}$$

$$\nabla\!\!\!/_L\underline{\beta} = {}^\star\mathcal{D}_1(\rho, \sigma) + r^{-1}\check{R} + A \cdot \tilde{R}, \tag{6.4}$$

where

$$\tilde{R} := R_0 + \nabla\!\!\!/ A + A \cdot \underline{A} + r^{-1}\underline{A}.$$

We will consider the commutators

$$C(\check{R}) = (C_1(\check{R}), C_2(\check{R}), C_3(\check{R})) \tag{6.5}$$

given in [2, Definition 6.3] which, by using the above conventions, can be written symbolically as

$$C_1(\check{R}) = \nabla\!\!\!/\mathcal{D}^{-1}[\mathcal{D}_t, \mathcal{D}^{-1}]\check{R},$$
$$C_2(\check{R}) = \nabla\!\!\!/[\mathcal{D}_t, \mathcal{D}^{-1}]\mathcal{D}^{-1}\check{R},$$
$$C_3(\check{R}) = [\mathcal{D}_t, \nabla\!\!\!/]\mathcal{D}^{-2}\check{R}.$$

Corresponding to (6.3) and (6.4), we introduce the error terms

$$Err := \mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma}) - an\beta \quad \text{and} \quad \widetilde{Err} := {}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta} - an(\rho, \sigma). \tag{6.6}$$

Denote by $\mathfrak{F}$ either Err or $\widetilde{Err}$. Symbolically, $\mathfrak{F}$ has the form

$$\mathfrak{F} = \mathcal{D}^{-1}\{an(r^{-1}\check{R} + A \cdot \tilde{R})\}.$$

We then infer from (6.3) and (6.4) the symbolic expression

$$\mathcal{D}^{-1}\mathcal{D}_t\check{R} = anR_0 + \mathfrak{F}. \tag{6.7}$$

[9] For simplicity, we use $(a\delta + 2a\lambda)$ to denote the pair of quantities $(a\delta + 2a\lambda, 0)$.

By using (4.15) with $\theta=0$ and $p=\frac{4}{3}$, Proposition 3.4 and the Hölder inequality we infer that

$$\|\mathfrak{F}\|_{\mathcal{P}^0} \lesssim \|r^{\frac{1}{2}} A\cdot \tilde{R}\|_{L_t^2 L_x^{\frac{4}{3}}} + \|\check{R}\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{6.8}$$

The purpose of this section is to prove

Proposition 6.1. *There hold the following decomposition for commutators,*

$$C(\check{R}) = \mathcal{D}_t P + E,$$
$$[\mathcal{D}_t, \nabla\!\!\!/ \mathcal{D}_1^{-1}](a\delta+2a\lambda) = \mathcal{D}_t P' + E',$$

where P, P' and E, E' are S tangent tensors verifying

$$\mathcal{N}_1(P) + \mathcal{N}_1(P') + \|E\|_{\mathcal{P}^0} + \|E'\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$
$$\lim_{t\to 0}\big(r\|P\|_{L^\infty(S)} + r\|P'\|_{L^\infty(S)}\big) = 0.$$

6.2. **Proof of Proposition 6.2: Part I.** In order to prove Proposition 6.1, let us consider the structure of commutators. We first use (6.1) to write

$$\big(C_2(\check{R}), \nabla\!\!\!/[\mathcal{D}_t, \mathcal{D}_1^{-1}](a\delta+2a\lambda)\big) = \nabla\!\!\!/[\mathcal{D}_t, \mathcal{D}^{-1}]_g \mathcal{F} + \nabla\!\!\!/ \mathcal{D}^{-1}(an\beta\cdot \mathcal{D}^{-1}\mathcal{F}), \tag{6.9}$$

$$\big(C_3(\check{R}), [\mathcal{D}_t, \nabla\!\!\!/]\mathcal{D}_1^{-1}(a\delta+2a\lambda)\big) = [\mathcal{D}_t, \nabla\!\!\!/]_g \mathcal{D}^{-1}\mathcal{F} + an\beta\cdot \mathcal{D}^{-1}\mathcal{F}, \tag{6.10}$$

where

$$[\mathcal{D}_t, \mathcal{D}^{-1}]_g \mathcal{F} := \mathcal{D}^{-1}\big(an(A+r^{-1})\cdot \nabla\!\!\!/ \mathcal{D}^{-1}\mathcal{F} + an(A+r^{-1})\cdot A\cdot \mathcal{D}^{-1}\mathcal{F}\big).$$

The terms $\nabla\!\!\!/[\mathcal{D}_t, \mathcal{D}^{-1}]_g\mathcal{F}$ and $[\mathcal{D}_t, \nabla\!\!\!/]_g\mathcal{D}^{-1}\mathcal{F}$ are the "good" parts in the corresponding commutators and will be proved to be $\mathcal{P}^0$ bounded (see (6.13)-(6.16)). The terms $\nabla\!\!\!/\mathcal{D}^{-1}(an\beta\cdot\mathcal{D}^{-1}\mathcal{F})$ and $an\beta\cdot\mathcal{D}^{-1}\mathcal{F}$ in (6.9) and (6.10) can not be bounded in $\mathcal{P}^0$ norm and will be further decomposed in Section 6.7.

Let us first establish (6.11), (6.13)-(6.16) in two steps in the following result

Proposition 6.2. *For the error terms $C_0(\check{R})$, $C_1(\check{R})$, $C_2(\check{R})$, $C_3(\check{R})$, etc, there hold*

$$\|C_0(\check{R})\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{6.11}$$

$$\|C_1(\check{R})\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{6.12}$$

$$C_2(\check{R}) = \nabla\!\!\!/\mathcal{D}^{-1}(an\beta\cdot\mathcal{D}^{-2}\check{R}) + err, \tag{6.13}$$

$$C_3(\check{R}) = an\beta\cdot\mathcal{D}^{-2}\check{R} + err \tag{6.14}$$

$$\nabla\!\!\!/[\mathcal{D}_t, \mathcal{D}_1^{-1}](a\delta+2a\lambda) = \nabla\!\!\!/\mathcal{D}^{-1}(an\beta\cdot\mathcal{D}^{-1}(a\delta+2a\lambda)) + err \tag{6.15}$$

$$[\mathcal{D}_t, \nabla\!\!\!/]\mathcal{D}_1^{-1}(a\delta+2a\lambda) = an\beta\cdot\mathcal{D}^{-1}(a\delta+2a\lambda) + err \tag{6.16}$$

with

$$\|err\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

We will rely on (2.79), (**SobM1**), (1.13), Proposition 2.7 and Remark 2.2, i.e.

$$\|\underline{A}\|_{L_x^4 L_t^\infty} + \|\underline{A}\|_{L^6} + \mathcal{N}_1(\underline{A}) + \|r^{\frac{1}{2}}\nabla\!\!\!/\mathrm{tr}\chi\|_{L_x^2 L_t^\infty} + \|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2} + \|R_0\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{6.17}$$

where ι, κ are regarded as elements of A.

Step 1. We first prove (6.11). In view of (6.1),

$$C_0(\check{R}) = \mathcal{D}^{-1}(an\{(A+r^{-1})(\nabla\!\!\!/\mathcal{D}^{-1}\check{R}) + (A+r^{-1})\cdot A\cdot\mathcal{D}^{-1}\check{R} + \beta\cdot\mathcal{D}^{-1}\check{R}\}). \tag{6.18}$$

Using (4.15) with $\theta = 0$ and $p = \frac{4}{3}$, Proposition 3.4, also with the help of (6.17) and Hölder inequality, we can estimate the various terms in (6.18) to get

$$\|C_0(\check{R})\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0 + \Delta_0 \cdot \mathcal{N}_1(\mathcal{D}^{-1}\check{R}). \tag{6.19}$$

By the definition of $\mathcal{N}_1(\mathcal{D}^{-1}\check{R})$ and Proposition 3.4 it follows that

$$\mathcal{N}_1(\mathcal{D}^{-1}\check{R}) \lesssim \mathcal{R}_0 + \Delta_0^2 + \|\mathcal{D}^{-1}\mathcal{D}_t\check{R}\|_{L_t^2 L_x^2} + \|C_0(\check{R})\|_{L_t^2 L_x^2}.$$

While it follows from (6.7), (6.8) and (1.13) that

$$\|\mathcal{D}^{-1}\mathcal{D}_t\check{R}\|_{L^2} \lesssim \|\mathfrak{F}\|_{\mathcal{P}^0} + \|anR_0\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Combining the above three inequalities and using the smallness of Δ_0 we obtain (6.11).

In the above proof, together with Lemma 3.1, (2.79) and Remark 2.1 we have also verified the following

Proposition 6.3.

$$\|\mathcal{D}^{-1}\mathcal{D}_t\check{R}\|_{L^2} \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{6.20}$$

$$\|[\mathcal{D}_t, \mathcal{D}^{-1}]\check{R}\|_{L^2} \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{6.21}$$

$$\mathcal{N}_1(\mathcal{F}) \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{6.22}$$

$$\mathcal{N}_1(\nabla\!\!\!/\mathcal{D}^{-1}\mathcal{F}) \lesssim \mathcal{R}_0 + \Delta_0^2, \qquad \mathcal{N}_2(\mathcal{D}^{-1}\mathcal{F}) \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{6.23}$$

where $\mathcal{F}$ denotes either $\mathcal{D}^{-1}\check{R}$ or $(a\delta + 2a\lambda)$.

Step 2. We will prove (6.13)-(6.16). Let us first establish the following

Lemma 6.1. *Denote by $\mathcal{D}^{-1}$ one of the operators among $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$ and ${}^\star\mathcal{D}_1^{-1}$. For any S_t tangent tensor H, $b > 4$*

$$\|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}(an\nabla\!\!\!/\mathcal{D}^{-1}H)\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(\mathcal{D}^{-1}H)\|\zeta + \underline{\zeta}\|_{L_t^\infty L_x^4} + \mathcal{N}_1(r^{-\frac{1}{2}}\mathcal{D}^{-1}H). \tag{6.24}$$

Proof. Using Propositions 3.4 and 4.3,

$$\begin{aligned}
&\|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}(an\nabla\!\!\!/\mathcal{D}^{-1}H)\|_{L_t^b L_x^2}\\
&\quad\lesssim \|r^{-1-\frac{1}{b}}\left(|\mathcal{D}^{-1}(\nabla\!\!\!/(an)\mathcal{D}^{-1}H)| + |\mathcal{D}^{-1}\nabla\!\!\!/(an\mathcal{D}^{-1}H)|\right)\|_{L_t^b L_x^2}\\
&\quad\lesssim \|r^{-\frac{1}{2}-\frac{1}{b}}\nabla\!\!\!/(an)\mathcal{D}^{-1}H\|_{L_t^b L_x^{4/3}} + \|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}H\|_{L_t^b L_x^2}.
\end{aligned}$$

Since by using (2.28), we obtain

$$\begin{aligned}
\|r^{-\frac{1}{2}-\frac{1}{b}}\nabla\!\!\!/(an)\mathcal{D}^{-1}H\|_{L_t^b L_x^{4/3}} &\lesssim \|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}^{-1}H\|_{L_t^b L_x^2}\|\nabla\!\!\!/(an)\|_{L_t^\infty L_x^4}\\
&\lesssim \mathcal{N}_1(\mathcal{D}^{-1}H)\|\zeta + \underline{\zeta}\|_{L_t^\infty L_x^4},
\end{aligned}$$

and $\|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}H\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(r^{-1/2}\mathcal{D}^{-1}H)$. Then (6.24) follows. $\square$

The proof of (6.13)-(6.16) can be completed by using (6.22) combined with the following result.

Lemma 6.2. *Denote by $\mathcal{D}^{-1}$ either $\mathcal{D}_1^{-1}$ or $\mathcal{D}_2^{-1}$. For appropriate S-tangent tensor field F, there hold*

$$\|[\mathcal{D}_t, \nabla\!\!\!/]_g\mathcal{D}^{-1}F\|_{\mathcal{P}^0} + \|\nabla\!\!\!/[\mathcal{D}^{-1}, \mathcal{D}_t]_g F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F), \tag{6.25}$$

$$\|[\mathcal{D}_t, \nabla\!\!\!/\mathcal{D}^{-1}]_g F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F). \tag{6.26}$$

Proof. Observe that

$$\|[\mathcal{D}_t, \nabla\!\!\!/ \mathcal{D}^{-1}]_g F\|_{\mathcal{P}^0} \lesssim \|[\mathcal{D}_t, \nabla\!\!\!/]_g \mathcal{D}^{-1} F\|_{\mathcal{P}^0} + \|\nabla\!\!\!/ [\mathcal{D}^{-1}, \mathcal{D}_t]_g F\|_{\mathcal{P}^0},$$

it suffices to prove (6.25) only.

We first derive in view of (6.1) by using Lemma 4.4 with $4 < b < \infty$,

$$\begin{aligned}\|[\mathcal{D}_t, \nabla\!\!\!/]_g \mathcal{D}^{-1} F\|_{\mathcal{P}^0} &\lesssim \mathcal{N}_1(\nabla\!\!\!/ \mathcal{D}^{-1} F)\left(\|r^{\frac{1}{2}} \nabla\!\!\!/ (anA)\|_{L_t^2 L_x^2} + \|r^{-\frac{1}{b}} anA\|_{L_t^b L_x^2}\right) \\ &\quad + \mathcal{N}_2(\mathcal{D}^{-1} F)\big(\|anA \cdot A\|_{\mathcal{P}^0} + \|r^{-\frac{1}{2}} anA\|_{\mathcal{P}^0}\big) &(6.27)\\ &\quad + \|r^{-1} an \nabla\!\!\!/ \mathcal{D}^{-1} F\|_{\mathcal{P}^0}. &(6.28)\end{aligned}$$

By (4.21) and (6.17), we obtain

$$\|r^{-1} anA\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \quad \|anA \cdot A\|_{\mathcal{P}^0} \lesssim \Delta_0 \mathcal{N}_1(A) \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Then the term in (6.27) can be bounded by $(\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(\mathcal{D}^{-1} F)$.

We then consider the term in (6.28). In view of (4.13) and Theorem 4.3,

$$\|r^{-1} an \nabla\!\!\!/ \mathcal{D}^{-1} F\|_{\mathcal{P}^0} \lesssim \|r^{-1} F\|_{\mathcal{P}^0} + \Delta_0 \|r^{-1} \mathcal{D}^{-1} F\|^q_{L_t^b L_x^2} \|r^{-1} F\|^{1-q}_{L^2(\mathcal{H})}.$$

Since by Proposition 4.3 and (**SobM1**), we obtain,

$$\|r^{-1} \mathcal{D}^{-1} F\|_{L_t^b L_x^2} \lesssim \|F\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(F),$$

also by using (4.23), we deduce

$$\|r^{-1} an \nabla\!\!\!/ \mathcal{D}^{-1} F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F) + \Delta_0 \mathcal{N}_1(F).$$

By (6.17), it is easy to check

$$\|r^{\frac{1}{2}} \nabla\!\!\!/ (anA)\|_{L_t^2 L_x^2} + \|r^{-\frac{1}{b}} anA\|_{L_t^b L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Consequently, we conclude that

$$\|[\mathcal{D}_t, \nabla\!\!\!/]_g \mathcal{D}^{-1} F\|_{\mathcal{P}^0} \lesssim (\Delta_0^2 + \mathcal{R}_0)(\mathcal{N}_1(\nabla\!\!\!/ \mathcal{D}^{-1} F) + \mathcal{N}_2(\mathcal{D}^{-1} F)) + \mathcal{N}_1(F). \tag{6.29}$$

We then infer from (6.29) and Lemma 3.1 that

$$\|[\mathcal{D}_t, \nabla\!\!\!/]_g \mathcal{D}^{-1} F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F). \tag{6.30}$$

Next, we prove for S-tangent tensor fields F on $\mathcal{H}$ the following inequality holds[10]

$$\|[\mathcal{D}_t, \mathcal{D}^{-1}]_g F\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(F) \quad \text{with } 4 < b < \infty. \tag{6.31}$$

Indeed, by using Proposition 4.3 with $p = 4/3$, Proposition 3.4, (**SobM1**), (6.17), (6.24) and Lemma 3.1 we then derive that

$$\begin{aligned}\|[\mathcal{D}_t, \mathcal{D}^{-1}]_g F\|_{L_t^b L_x^2} &\lesssim \|r^{1/2} A \cdot \nabla\!\!\!/ \mathcal{D}^{-1} F\|_{L_t^b L_x^{4/3}} + \|r^{1/2} A \cdot A \cdot \mathcal{D}^{-1} F\|_{L_t^b L_x^{4/3}} \\ &\quad + \|r^{-1} \mathcal{D}^{-1}(an \nabla\!\!\!/ \mathcal{D}^{-1} F)\|_{L_t^b L_x^2} + \|r^{-1/2} A \cdot \mathcal{D}^{-1} F\|_{L_t^b L_x^{4/3}} \\ &\lesssim \|A\|_{L_t^b L_x^2} \|\nabla\!\!\!/ \mathcal{D}^{-1} F\|_{L_t^\infty L_x^4} \\ &\quad + \|\mathcal{D}^{-1} F\|_{L_t^\infty L_x^4} \times (\|A \cdot A\|_{L_t^b L_x^2} + \|\|r^{-\frac{1}{2}} A\|_{L_t^b L_x^2}) \\ &\quad + \mathcal{N}_1(r^{-1/2} \mathcal{D}^{-1} F) + \Delta_0 \mathcal{N}_1(\mathcal{D}^{-1} F) \\ &\lesssim \mathcal{N}_1(\nabla\!\!\!/ \mathcal{D}^{-1} F) + \mathcal{N}_2(\mathcal{D}^{-1} F) \lesssim \mathcal{N}_1(F).\end{aligned}$$

[10] We will improve the right hand side of (6.31) to be $\mathcal{N}_1(\mathcal{D}^{-1} F)$ in the next section.

The combination of (6.31), (6.30) and (4.16) gives

$$\begin{aligned}\|\nabla\!\!\!\!/[\mathcal{D}^{-1},\mathcal{D}_t]_g F\|_{\mathcal{P}^0} &\lesssim \|[\mathcal{D}_t,\nabla\!\!\!\!/]_g\mathcal{D}^{-1}F\|_{\mathcal{P}^0}+\Delta_0\|[\mathcal{D}_t,\mathcal{D}^{-1}]_g F\|^q_{L_t^b L_x^2}\cdot\|[\mathcal{D}_t,\nabla\!\!\!\!/]_g\mathcal{D}^{-1}F\|_{L^2}^{1-q}\\ &\lesssim \mathcal{N}_1(F).\end{aligned}$$

where in the above inequalities, $b>4$ and $\alpha_0<q<1$. □

6.3. **Proof of Proposition 6.2: Part II.** We record the following result, which, for completeness, will be proved in Appendix by using Lemma 8.1.

Lemma 6.3. *For any smooth S-tangent tensor field F and for exponent $2\le q\le\infty$, we have the following inequality for $k>0$*

$$\|r^{-\frac12-\frac1q}P_k F\|_{L_t^q L_x^2}\lesssim 2^{-\frac12 k-\frac1q k}\mathcal{N}_1(F), \tag{6.32}$$

$$\|r^{-\frac1q}F_k\|_{L_t^q L_x^4}\lesssim 2^{-\frac{k}{q}}\mathcal{N}_1(F) \tag{6.33}$$

We will complete the proof of Proposition 6.2 by studying error type terms in Proposition 6.5

Let us first establish the following result with the help of Lemma 6.3.

Proposition 6.4. *Let $\mathcal{D}^{-1}$ denote either $\mathcal{D}_1^{-1}$ or ${}^\star\mathcal{D}_1^{-1}$. For any S-tangent tensor fields F and G on $\mathcal{H}$ there holds*

$$\|r^{-\frac1b}\mathcal{D}^{-1}(anF\cdot\nabla\!\!\!\!/ G)\|_{L_t^b L_x^2}\lesssim\mathcal{N}_1(F)\mathcal{N}_1(G),\quad \textit{with } 4<b<\infty.$$

Proof. Note that based on Lemma 8.1 in Appendix, the inequalities [11, Lemma 5.1,(6.25),(6.26)] still hold true. We only need to modify the case $l<n<m$ of the proof in [11, P.310-311].

When $l<n<m$,

$$anF_n\cdot\nabla\!\!\!\!/ G_m=\nabla\!\!\!\!/(anF_n\cdot G_m)-\nabla\!\!\!\!/(an)F_n\cdot G_m-an\nabla\!\!\!\!/ F_n\cdot G_m,$$

thus we need to consider the three terms

$$\begin{aligned}\mathcal{I}^1_{lnm}&:=\|r^{-\frac1b}P_l\mathcal{D}^{-1}(an\nabla\!\!\!\!/ F_n\cdot G_m)\|_{L_t^b L_x^2}\\ \mathcal{I}^2_{lnm}&:=\|r^{-\frac1b}P_l\mathcal{D}^{-1}\nabla\!\!\!\!/(anF_n\cdot G_m)\|_{L_t^b L_x^2}\\ \mathcal{I}^3_{lnm}&:=\|r^{-\frac1b}P_l\mathcal{D}^{-1}(\nabla\!\!\!\!/(an)F_n\cdot G_m)\|_{L_t^b L_x^2}.\end{aligned}$$

Using $C^{-1}<an<C$, following the same procedure in [11], we can get

$$\sum_{0<l<n<m}(\mathcal{I}^1_{lnm}+\mathcal{I}^2_{lnm})\lesssim\mathcal{N}_1(F)\mathcal{N}_1(G). \tag{6.34}$$

Now consider $\mathcal{I}^3_{lnm}$.

By Lemma 4.3 with $p=\frac43$, Proposition 4.1 (i) followed with (**SobM1**), and (6.33),

$$\begin{aligned}\mathcal{I}^3_{lnm}&=\|r^{-\frac1b}P_l\mathcal{D}^{-1}(an(\zeta+\underline{\zeta})F_n\cdot G_m)\|_{L_t^b L_x^2(\mathcal{H})}\\ &\lesssim 2^{-\frac{l}{2}}\|r^{\frac12-\frac1b}(\zeta+\underline{\zeta})anF_n\cdot G_m\|_{L_t^b L_x^{4/3}}\\ &\lesssim 2^{-\frac{l}{2}}\|r^{\frac12-\frac1b}F_n\cdot G_m\|_{L_t^b L_x^2}\|A\|_{L_t^\infty L_x^4}\\ &\lesssim 2^{-\frac{l}{2}}\|r^{-\frac1b+\frac12}G_m\|_{L_t^b L_x^4}\|F_n\|_{L_t^\infty L_x^4}\\ &\lesssim 2^{-\frac{l}{2}-\frac{m}{b}}\mathcal{N}_1(G)\cdot\mathcal{N}_1(F),\end{aligned}$$

we obtain

$$\sum_{0<l<n<m} \mathcal{I}^3_{lnm} \lesssim \mathcal{N}_1(G)\mathcal{N}_1(F).$$

The proof is therefore complete. □

Let $\mathcal{D}$ be one of the operators $\mathcal{D}_1, {}^\star\mathcal{D}_1$ or $\mathcal{D}_2$. In the following result, we use Proposition 6.4 to estimate the error type terms.

Proposition 6.5. *For S-tangent tensors G on $\mathcal{H}$ verifying $\mathcal{N}_1(G) < \infty$, set*

$$\mathcal{E}_1(G) := r^{-1}\mathcal{D}^{-1}(anA \cdot G) \ \text{ or } \ \mathcal{D}^{-1}(anA \cdot A \cdot G),$$
$$\mathcal{E}_2(G) := \mathcal{D}^{-1}(an\nabla\!\!\!\!/ A \cdot G) \ \text{ or } \ \mathcal{D}^{-1}(an\nabla\!\!\!\!/ G \cdot A),$$

The following estimates hold

$$\|r^{-\frac{1}{b}}\mathcal{E}_1(G)\|_{L^b_t L^2_x} + \|r^{-\frac{1}{b}}\mathcal{E}_2(G)\|_{L^b_t L^2_x} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(G)$$

where $4 < b < \infty$.

Proof. Using Proposition 4.3 with $p = \frac{4}{3}$, (**SobM1**) and (6.17), we get

$$\begin{aligned}\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(anA \cdot A \cdot G)\|_{L^b_t L^2_x} &\lesssim \|r^{\frac{1}{2}-\frac{1}{b}}A \cdot A \cdot G\|_{L^b_t L^{4/3}_x} \lesssim \|A\|^2_{L^\infty_t L^4_x}\|G\|_{L^b_t L^4_x} \\ &\lesssim (\Delta_0^2 + \mathcal{R}_0)^2\mathcal{N}_1(G),\end{aligned}$$

by using Proposition 3.4, (2.28), (6.17) and (**SobM1**), we have

$$\begin{aligned}\|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}(anA \cdot G)\|_{L^b_t L^2_x} &\lesssim \|r^{-\frac{1}{b}}A \cdot G\|_{L^b_t L^2_x} \lesssim \|r^{-\frac{1}{b}}A\|_{L^b_t L^4_x}\|G\|_{L^\infty_t L^4_x} \\ &\lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(G).\end{aligned}$$

For $\mathcal{E}_2(G)$, we infer from Proposition 6.4 and (6.17) that

$$\|r^{-\frac{1}{b}}\mathcal{E}_2(G)\|_{L^b_t L^2_x} \lesssim \mathcal{N}_1(G)\mathcal{N}_1(A) \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(G).$$

We thus obtain the desired estimates. □

By analyzing the expression of β and $C_0(F) := [\mathcal{D}_t, \mathcal{D}^{-1}]F$, we have

Corollary 2. *The following inequalities hold for any S-tangent tensor F,*

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(an\beta \cdot F)\|_{L^b_t L^2_x} \lesssim \mathcal{N}_1(F)(\Delta_0^2 + \mathcal{R}_0) \tag{6.35}$$

$$\|r^{-\frac{1}{b}}C_0(F)\|_{L^b_t L^2_x} \lesssim \mathcal{N}_1(r^{-\frac{1}{2}}\mathcal{D}^{-1}F) \tag{6.36}$$

where $4 < b < \infty$.

Proof. Using Codazzi equation (2.6), i.e.

$$an\beta = an\nabla\!\!\!\!/ A + an(A \cdot A + r^{-1}A),$$

we infer

$$\mathcal{D}^{-1}(an\beta \cdot F) = \mathcal{E}_1(F) + \mathcal{E}_2(F).$$

Whence (6.35) follows from Proposition 6.5.

Similarly, using (6.1) we can write

$$C_0(F) = \mathcal{E}_1(\mathcal{D}^{-1}F) + \mathcal{E}_2(\mathcal{D}^{-1}F) + r^{-1}\mathcal{D}^{-1}(an\nabla\!\!\!\!/ \mathcal{D}^{-1}F).$$

For the last term, using (6.24) with $H = F$, we infer

$$\|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}(an\nabla\!\!\!\!/ \mathcal{D}^{-1}F)\|_{L^b_t L^2_x} \lesssim \mathcal{N}_1(r^{-\frac{1}{2}}\mathcal{D}^{-1}F)(\Delta_0^2 + \mathcal{R}_0 + 1).$$

The desired estimate then follows from Proposition 6.5.

□

Proof of (6.12) in Proposition 6.2. Combining Proposition 4.3, (6.36) and (6.22) we derive for $\mathcal{D}^{-1}$ either $\mathcal{D}_2^{-1}$ or $\mathcal{D}_1^{-1}$, $4<b<\infty$

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}C_0(\check{R})\|_{L_t^b L_x^2}\lesssim \|r^{1-\frac{1}{b}}C_0(\check{R})\|_{L_t^b L_x^2}\lesssim \mathcal{N}_1(r^{\frac{1}{2}}\mathcal{D}^{-1}\check{R})\lesssim \Delta_0^2+\mathcal{R}_0. \tag{6.37}$$

Observe that $C_1(\check{R})$ can be written symbolically in the form

$$C_1(\check{R})=\nabla\!\!\!\!/\,\mathcal{D}^{-1}C_0(\check{R}).$$

By Hodge-elliptic estimate (4.16) with $\frac{1}{2}<q<1$ and $4<b<\infty$, (6.11), (6.21) and (6.37)

$$\begin{aligned}\|C_1(\check{R})\|_{\mathcal{P}^0}&\lesssim \|C_0(\check{R})\|_{\mathcal{P}^0}+\Delta_0\|\mathcal{D}^{-1}C_0(\check{R})\|^q_{L_t^b L_x^2}\|C_0(\check{R})\|^{1-q}_{L^2(\mathcal{H})}\\&\lesssim \Delta_0^2+\mathcal{R}_0+\Delta_0\|\mathcal{D}^{-1}C_0(\check{R})\|^q_{L_t^b L_x^2}(\Delta_0^2+\mathcal{R}_0)^{1-q}\lesssim \Delta_0^2+\mathcal{R}_0.\end{aligned}$$

This is the desired estimate. $\square$

6.4. **A preliminary estimate for (ρ,σ).** Define

$$\bar{\rho}=\frac{1}{|S_t|}\int_{S_t}\rho d\mu_\gamma \quad\text{and}\quad \bar{\sigma}=\frac{1}{|S_t|}\int_{S_t}\sigma d\mu_\gamma.$$

We have

Lemma 6.4.

$$|r^{\frac{3}{2}}\bar{\rho}|+|r^{\frac{3}{2}}\bar{\sigma}|\lesssim \Delta_0^2+\mathcal{R}_0, \tag{6.38}$$

$$|r^{\frac{3}{2}}\bar{\bar{\rho}}|+|r^{\frac{3}{2}}\bar{\bar{\sigma}}|\lesssim \Delta_0^2+\mathcal{R}_0. \tag{6.39}$$

Proof. By [2, Eq. (41)], i.e.

$$\frac{d}{ds}\rho+\frac{3}{2}\mathrm{tr}\chi\rho=F \tag{6.40}$$

where $F=\mathrm{div}\beta-\frac{1}{2}\hat{\underline{\chi}}\cdot\alpha+(\zeta+2\underline{\zeta})\cdot\beta$, the transport equation for $\bar{\rho}$ can be obtained as follows

$$\begin{aligned}\frac{d}{ds}(\bar{\rho})&=\nabla\!\!\!\!/_L(r^{-2}\int_S\rho)\\&=-\overline{an\mathrm{tr}\chi}(an)^{-1}\bar{\rho}+r^{-2}(an)^{-1}\int_S\left(\nabla\!\!\!\!/_L\rho+\mathrm{tr}\chi\rho\right)and\mu_\gamma\\&=-(an)^{-1}\overline{an\mathrm{tr}\chi}\bar{\rho}+(an)^{-1}\overline{\left(-\frac{1}{2}an\mathrm{tr}\chi\rho+anF\right)},\end{aligned}$$

and

$$\begin{aligned}\frac{d}{ds}(r^3\bar{\rho})&=\frac{3r^3}{2}\overline{an\mathrm{tr}\chi}(an)^{-1}\bar{\rho}\\&\quad+r^3\left\{-(an)^{-1}\overline{an\mathrm{tr}\chi}\bar{\rho}+(an)^{-1}\overline{\left(-\frac{1}{2}an\mathrm{tr}\chi\rho+anF\right)}\right\}\\&=-\frac{1}{2}r^3(an)^{-1}\overline{an\mathrm{tr}\chi(\rho-\bar{\rho})}+r^3(an)^{-1}\overline{anF}.\end{aligned}$$

Integrating the above identity in t, due to $\lim_{t\to0}r^\theta\rho=0$ for any $\theta>0$, we can obtain

$$r^{\frac{3}{2}}\bar{\rho}(t)=r^{-\frac{3}{2}}\int_0^t r^3\left(-\frac{1}{2}\overline{an\mathrm{tr}\chi\cdot\mathcal{O}sc(\rho)}+\overline{anF}\right)dt'.$$

In view of (2.23), (1.13) and Proposition 2.2, we obtain

$$r^{-\frac{3}{2}}\int_0^t \left|r^3\overline{an\mathrm{tr}\chi\cdot\mathcal{O}sc(\rho)}\right| \le \|r\mathcal{O}sc(\rho)\|_{L_t^2L_\omega^2}\|r'\|_{L^2(0,t]}r^{-\frac{3}{2}} \lesssim \|\rho\|_{L^2(\mathcal{H})} \lesssim \mathcal{R}_0.$$

By integration by part on $S=S_t$, we can obtain

$$\begin{aligned} r^2\overline{anF} &= \int_S \left(an(\mathrm{div}\beta - \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha) + an(\zeta+2\underline{\zeta})\cdot\beta\right)d\mu_\gamma \\ &= \int_S an(\underline{\zeta}\beta - \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha)d\mu_\gamma = \int_S an\underline{A}\cdot R_0 d\mu_\gamma. \end{aligned}$$

Hence by (1.13) and Proposition 2.3,

$$\begin{aligned} \left|r^{-\frac{3}{2}}\int_0^t {r'}^3\overline{anF}dt'\right| &\le r^{-\frac{3}{2}}\|r'\|_{L^2(0,t]}\|(r')^2\underline{A}\cdot R_0\|_{L_t^2L_\omega^1} \\ &\le \|R_0\|_{L^2}\|r\underline{A}\|_{L_t^\infty L_\omega^2} \\ &\lesssim \Delta_0^2+\mathcal{R}_0. \end{aligned}$$

Following the same procedure as above, we can obtain the same estimate for $\bar{\sigma}$ in view of [2, Eq. (42)].

Note that

$$|r(\overline{\hat{\chi}\cdot\underline{\hat{\chi}}}, \overline{\hat{\chi}\wedge\underline{\hat{\chi}}})| \lesssim \|r^{\frac{1}{2}}\underline{A}\|^2_{L_t^\infty L_\omega^2} \lesssim (\Delta_0^2+\mathcal{R}_0)^2 \tag{6.41}$$

(6.39) follows by connecting (6.38) with (6.41).

□

Proposition 6.6. *For $4<b<\infty$, there hold*

$$\|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}(an(\check{\rho},-\check{\sigma}))\|_{L_t^bL_x^2} \lesssim \Delta_0^2+\mathcal{R}_0, \tag{6.42}$$

$$\|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}(an(\rho,-\sigma))\|_{L_t^bL_x^2} \lesssim \Delta_0^2+\mathcal{R}_0. \tag{6.43}$$

Proof. Let $H=(\check{\rho}-\bar{\check{\rho}}, -\check{\sigma}+\bar{\check{\sigma}})$. In view of $\mathcal{D}_1\mathcal{D}_1^{-1}H=H$,

$$\mathcal{D}_1^{-1}(an(\check{\rho},-\check{\sigma})) = \mathcal{D}_1^{-1}(an\mathcal{D}_1\mathcal{D}_1^{-1}H) + \mathcal{D}_1^{-1}(an(\bar{\check{\rho}},-\bar{\check{\sigma}})).$$

By Proposition 4.3,

$$\|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}(an(\bar{\check{\rho}},-\bar{\check{\sigma}}))\|_{L_t^bL_x^2} \lesssim \|r^{\frac{3}{2}-\frac{1}{b}}an(\bar{\check{\rho}},-\bar{\check{\sigma}})\|_{L_t^bL_\omega^2}.$$

Hence, in view of (6.39) and (1.13)

$$\begin{aligned} \|r^{\frac{3}{2}-\frac{1}{b}}(an(\bar{\check{\rho}},-\bar{\check{\sigma}}))\|_{L_t^bL_\omega^2} &\lesssim \|r^{\frac{3}{2}}an(\bar{\check{\rho}},-\bar{\check{\sigma}})\|^{1-\frac{2}{b}}_{L_t^\infty L_\omega^\infty}\|ran(\bar{\check{\rho}},-\bar{\check{\sigma}})\|^{\frac{2}{b}}_{L_t^2L_\omega^2} \\ &\lesssim \Delta_0^2+\mathcal{R}_0. \end{aligned} \tag{6.44}$$

By Leibnitz rule, Proposition 4.3 and (2.28), we have

$$\begin{aligned} \|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}(an\mathcal{D}_1\mathcal{D}_1^{-1}H)\|_{L_t^bL_x^2} &\lesssim \|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}\mathcal{D}_1(an\mathcal{D}_1^{-1}H)\|_{L_t^bL_x^2} \\ &\quad + \|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}(\nabla\!\!\!/(an)\mathcal{D}_1^{-1}H)\|_{L_t^bL_x^2} \\ &\lesssim \|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}H\|_{L_t^bL_x^2} + \|r^{-\frac{1}{b}}(\zeta+\underline{\zeta})\mathcal{D}_1^{-1}H\|_{L_t^bL_x^{4/3}} \\ &\lesssim \mathcal{N}_1(\mathcal{D}_1^{-1}H) + \|r^{-\frac{1}{b}}\mathcal{D}_1^{-1}H\|_{L_t^bL_x^2}\|\zeta+\underline{\zeta}\|_{L_t^\infty L_x^4} \\ &\lesssim \mathcal{N}_1(\mathcal{D}_1^{-1}H)(1+\Delta_0^2+\mathcal{R}_0) \lesssim \Delta_0^2+\mathcal{R}_0 \end{aligned}$$

For the last two inequalities, we employed (6.17) and (6.22).

(6.43) can be obtained by combining (6.42) with the estimate for $r^{-1}\mathcal{D}^{-1}(anA\cdot\underline{A})$ in view of the estimate for $\mathcal{E}_1(\underline{A})$ in Proposition 6.5. $\square$

6.5. **$L_t^b L_x^2$ estimates for $\mathcal{D}^{-1}E_1^G$.** For arbitrary S-tangent tensor field F, we denote by E_1^G either $[\mathcal{D}_t, \mathcal{D}_1^{-1}](\check{\rho}, -\check{\sigma})\cdot F$ or $Err\cdot F$. In what follows, we establish estimates for $\|\mathcal{D}^{-1}E_1^G\|_{L_t^b L_x^2}$ with $4<b<\infty$, which will be employed for the Hodge-elliptic $\mathcal{P}^0$ estimates of error terms arising in the decomposition procedure in Section 6.7.

Proposition 6.7. *Denote by $\mathcal{D}$ either $\mathcal{D}_1$ or $\mathcal{D}_2$, for appropriate S-tangent tensor fields F, the following estimates hold*

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}E_1^G\|_{L_t^b L_x^2} \lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(F). \tag{6.45}$$

More precisely,

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(Err\cdot F)\|_{L_t^b L_x^2} \lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(F) \tag{6.46}$$

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}([\mathcal{D}_t, \mathcal{D}_1^{-1}](\check{\rho}, -\check{\sigma})\cdot F)\|_{L_t^b L_x^2} \lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(F). \tag{6.47}$$

where Err is defined in (6.6) and $4<b<\infty$.

In order to prove Proposition 6.7, we may use the error type terms introduced in Proposition 6.5 to rewrite (6.6) in view of (2.11) and (2.12) as

$$Err = \mathcal{D}_1^{-1}(an\operatorname{tr}\chi(\check{\rho}, -\check{\sigma})) + \mathcal{E}_1(\underline{A}) + \mathcal{E}_2(\underline{A}). \tag{6.48}$$

Proof of Proposition 6.7. (6.47) can be obtained by using Proposition 4.3, (6.36), (6.22) and (**SobM1**) as follows,

$$\begin{aligned}
\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(C_0(\check{R})\cdot F)\|_{L_t^b L_x^2} &\lesssim \|r^{\frac{1}{2}-\frac{1}{b}}C_0(\check{R})\|_{L_t^b L_x^2}\|F\|_{L_t^\infty L_x^4}\\
&\lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(F),
\end{aligned}$$

and similarly, by using Proposition 6.5, for $i=1,2$

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(\mathcal{E}_i(\underline{A})\cdot F)\|_{L_t^b L_x^2} \lesssim \|r^{\frac{1}{2}-\frac{1}{b}}\mathcal{E}_i(\underline{A})\|_{L_t^b L_x^2}\|F\|_{L_t^\infty L_x^4} \lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(F).$$

Thus, in order to prove (6.46), in view of (6.48), it remains to show

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\big(\mathcal{D}_1^{-1}(an\operatorname{tr}\chi(\check{\rho}, -\check{\sigma}))\cdot F\big)\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(F)\Delta_0.$$

For this estimate, we proceed as follows. Let $H=(\check{\rho}-\bar{\check{\rho}}, -\check{\sigma}+\bar{\check{\sigma}})$, then $H=\mathcal{D}_1\mathcal{D}_1^{-1}H$.

$$\begin{aligned}
\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\big(\mathcal{D}_1^{-1}(an\operatorname{tr}\chi(\check{\rho}, -\check{\sigma}))\cdot F\big)\|_{L_t^b L_x^2} &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(\mathcal{D}_1^{-1}(an\operatorname{tr}\chi\mathcal{D}_1\mathcal{D}_1^{-1}H)F)\|_{L_t^b L_x^2}\\
&\quad + \|r^{-\frac{1}{b}}\mathcal{D}^{-1}\big(\mathcal{D}_1^{-1}(an\operatorname{tr}\chi(\bar{\check{\rho}}, -\bar{\check{\sigma}}))\cdot F\big)\|_{L_t^b L_x^2}
\end{aligned}$$

By I_1 and I_2, we denote the two terms on the right of the inequality. Using Proposition 4.3, (**SobM1**), (2.28) and (6.22),

$$\begin{aligned}
I_1 &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(an\operatorname{tr}\chi\mathcal{D}^{-1}H\cdot F)\|_{L_t^b L_x^2} + \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(\mathcal{D}_1^{-1}(\nabla\!\!\!/(an\operatorname{tr}\chi)\mathcal{D}_1^{-1}H)\cdot F)\|_{L_t^b L_x^2}\\
&\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}H\cdot F\|_{L_t^b L_x^2} + \|r^{-\frac{1}{b}+\frac{1}{2}}\mathcal{D}_1^{-1}(\nabla\!\!\!/(an\operatorname{tr}\chi)\mathcal{D}^{-1}H)\|_{L_t^b L_x^2}\|F\|_{L_t^\infty L_x^4}\\
&\lesssim \|F\|_{L_t^\infty L_x^4}\|r^{-\frac{1}{b}}\mathcal{D}^{-1}H\|_{L_t^b L_x^4} + \|r^{1-\frac{1}{b}}\nabla\!\!\!/(an\operatorname{tr}\chi)\|_{L_t^b L_x^2}\|\mathcal{D}^{-1}H\|_{L_t^\infty L_x^4}\|F\|_{L_t^\infty L_x^4}\\
&\lesssim \mathcal{N}_1(F)\mathcal{N}_1(\mathcal{D}^{-1}H) \lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(F)
\end{aligned}$$

where we employed

$$\|r^{\frac{1}{2}-\frac{1}{b}}\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L_t^bL_x^2}\lesssim\|r^{\frac{1}{2}}\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L_t^\infty L_x^2}^{1-\frac{2}{b}}\|\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L^2}^{\frac{2}{b}}\lesssim\Delta_0^2+\mathcal{R}_0.$$

By Proposition 4.3, (6.44) and (**SobM1**),

$$\begin{aligned}I_2&\lesssim\|r^{-\frac{1}{b}+\frac{1}{2}}\mathcal{D}^{-1}(an\mathrm{tr}\chi(\bar{\check{\rho}},-\bar{\check{\sigma}}))\|_{L_t^bL_x^2}\|F\|_{L_t^\infty L_x^4}\\&\lesssim\|r^{\frac{3}{2}-\frac{1}{b}}an\mathrm{tr}\chi(\bar{\check{\rho}},-\bar{\check{\sigma}})\|_{L_t^bL_x^2}\|F\|_{L_t^\infty L_x^4}\\&\lesssim\|r^{\frac{3}{2}-\frac{1}{b}}(\bar{\check{\rho}},-\bar{\check{\sigma}})\|_{L_t^bL_\omega^2}\mathcal{N}_1(F)\\&\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(F).\end{aligned}$$

The proof is complete. □

6.6. $L_t^bL_x^2$ **estimates for** $\nabla\!\!\!/_L\mathcal{D}^{-1}\mathcal{F}$**.** We will establish the following

Proposition 6.8. *Denote by* $\mathcal{D}^{-1}\mathcal{F}$ *either* $\mathcal{D}^{-2}\check{R}$ *or* $\mathcal{D}_1^{-1}(a\delta+2a\lambda)$*. There holds*

$$\|r^{-\frac{1}{b}}\mathcal{D}_t\mathcal{D}^{-1}\mathcal{F}\|_{L_t^bL_x^2}\lesssim\mathcal{R}_0+\Delta_0^2,\quad 4<b<\infty.$$

Case 1: $\mathcal{F}=\mathcal{D}^{-1}\check{R}$**.**

We denote by $\mathcal{D}^{-1}\mathfrak{F}$ either $\mathcal{D}_2^{-1}Err$ or ${}^\star\mathcal{D}_1^{-1}\widetilde{Err}$. To prove Proposition 6.8, we will rely on (6.49) in the following result.

Proposition 6.9. *For* $\mathfrak{F}=(Err,\widetilde{Err})$ *with* Err *and* $\widetilde{Err}$ *given by (6.6), there hold*

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\mathfrak{F}\|_{L_t^bL_x^2}\lesssim\Delta_0^2+\mathcal{R}_0,\quad 4<b<\infty,\tag{6.49}$$

$$\|\nabla\!\!\!/\mathcal{D}^{-1}\mathfrak{F}\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0.\tag{6.50}$$

Assuming (6.49), now we prove Proposition 6.8.

Proof of Proposition 6.8 for Case 1. In view of the formula

$$\mathcal{D}_t\mathcal{D}^{-2}\check{R}=[\mathcal{D}_t,\mathcal{D}^{-1}]\mathcal{D}^{-1}\check{R}+\mathcal{D}^{-1}[\mathcal{D}_t,\mathcal{D}^{-1}]\check{R}+\mathcal{D}^{-2}\mathcal{D}_t\check{R},$$

we only need to show for $4<b<\infty$, there hold

$$\|r^{-\frac{1}{b}}[\mathcal{D}_t,\mathcal{D}^{-1}]\mathcal{D}^{-1}\check{R}\|_{L_t^bL_x^2}\lesssim\Delta_0^2+\mathcal{R}_0\tag{6.51}$$

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}[\mathcal{D}_t,\mathcal{D}^{-1}]\check{R}\|_{L_t^bL_x^2}\lesssim\Delta_0^2+\mathcal{R}_0\tag{6.52}$$

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-2}\mathcal{D}_t\check{R}\|_{L_t^bL_x^2}\lesssim\Delta_0^2+\mathcal{R}_0.\tag{6.53}$$

(6.51) follows from (6.36) with $F=\mathcal{D}^{-1}\check{R}$, by using the fact that $\mathcal{N}_1(r^{-\frac{1}{2}}\mathcal{D}^{-2}\check{R})\lesssim\mathcal{N}_2(\mathcal{D}^{-2}\check{R})\lesssim\Delta_0^2+\mathcal{R}_0$. (6.52) was proved in (6.37).

It only remains to prove (6.53). Consider first the case $\mathcal{D}^{-2}\mathcal{D}_t\check{R}=\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho},-\check{\sigma})$. By $an\beta=\nabla\!\!\!/(anA)+an(A\cdot A+r^{-1}A)$, (**SobM1**) and (6.17),

$$\begin{aligned}\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(an\beta)\|_{L_t^bL_x^2}&\lesssim\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(\nabla\!\!\!/(anA)+an(A\cdot A+r^{-1}A))\|_{L_t^bL_x^2}\\&\lesssim\|r^{-\frac{1}{b}}A\|_{L_t^bL_x^2}+\|r^{-\frac{1}{b}+1}A\cdot A\|_{L_t^bL_x^2}\\&\lesssim\mathcal{N}_1(A)+\mathcal{N}_1(A)^2\lesssim\Delta_0^2+\mathcal{R}_0.\end{aligned}$$

Then by (6.49), we obtain

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(Err+an\beta)\|_{L_t^bL_x^2}\lesssim\Delta_0^2+\mathcal{R}_0.$$

In view of the definition of Err in (6.6), we have

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-2}\mathcal{D}_t(\check{\rho},-\check{\sigma})\|_{L_t^b L_x^2} \lesssim \Delta_0^2+\mathcal{R}_0. \tag{6.54}$$

Using ${}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta} = an(\rho,\sigma)+\widetilde{Err}$, Proposition 6.6 and (6.49),

$$\begin{aligned}\|r^{-\frac{1}{b}}\mathcal{D}^{-2}\mathcal{D}_t\underline{\beta}\|_{L_t^b L_x^2} &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}_1^{-1}(an(\rho,\sigma))\|_{L_t^b L_x^2}+\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\mathfrak{F}\|_{L_t^b L_x^2}\\ &\lesssim \Delta_0^2+\mathcal{R}_0\end{aligned} \tag{6.55}$$

In view of (6.54) and (6.55), (6.53) is proved.

□

To prove Proposition 6.9, we will rely on the following result.

Lemma 6.5. *Let $\mathcal{D}^{-1}$ denote one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$ or ${}^\star\mathcal{D}_1^{-1}$. For any appropriate S-tangent tensor field G, there holds*

$$\|\mathcal{D}^{-1}(an\check{\rho}\cdot G)\|_{L_t^b L_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\mathcal{N}_1(G) \tag{6.56}$$

where $\alpha_0 \geq \frac{1}{2}$ and $4<b<\infty$.

Proof. We can adapt the proof for [11, Lemma 4.4] in view of $\|\nabla\!\!\!/(an)\|_{L_x^4 L_t^\infty} \lesssim \Delta_0^2+\mathcal{R}_0$ and $an<C$, to derive the following estimates for S tangent tensor fields F,

$$\|\Lambda^{-\alpha}(an\check{\rho}\cdot F_m)\|_{L^2(S)} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}2^m r^{-1}\|P_m F\|_{L^2(S)}, \tag{6.57}$$

$$\|P_m(an\check{\rho}\cdot\mathcal{D}^{-1}P_l F)\|_{L^2(S)} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}2^{\alpha m}r^{-\alpha_0}\|P_l F\|_{L^2(S)}, \tag{6.58}$$

where $\alpha>\alpha_0\geq 1/2$ and $\mathcal{D}^{-1}$ denotes either $\mathcal{D}_1^{-1}$ or ${}^\star\mathcal{D}_1^{-1}$.

Set $\Omega_{nl} := \mathcal{D}^{-1}P_l^2(an\check{\rho}\cdot P_n^2 G)$, with $l,n\in\mathbb{N}$. We now prove

$$\sum_{l,n>0}\|\Omega_{nl}\|_{L_t^b L_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\mathcal{N}_1(G), \tag{6.59}$$

and lower frequency terms can be treated similarly. By duality argument, Proposition 3.1 (iii) and Lemma 4.3,

$$\|\mathcal{D}^{-1}\Lambda^\alpha P_l F\|_{L^2(S)} \lesssim 2^{(-1+\alpha)l}r^{1-\alpha}\|F\|_{L^2(S)}. \tag{6.60}$$

We first prove (6.59) for the case $0<n<l$. With the help of (6.60) and (6.57),

$$\|\Omega_{nl}\|_{L_x^2} \lesssim 2^{-(1-\alpha)l}2^n r^{-\alpha}\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\|P_n G\|_{L_x^2}.$$

Take L_t^b norm for $4<b<\infty$, and (6.32) in Lemma 6.3

$$\|\Omega_{nl}\|_{L_t^b L_x^2} \lesssim 2^{-(1-\alpha)l}2^{n(\frac{1}{2}-\frac{1}{b})}r^{\frac{1}{2}+\frac{1}{b}-\alpha}\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\mathcal{N}_1(G).$$

Since we can choose $\alpha_0<\alpha<\frac{1}{2}+\frac{1}{b}$, we deduce

$$\sum_{0<n<l}\|\Omega_{nl}\|_{L_t^b L_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\mathcal{N}_1(G).$$

For the case $0<l<n$, let us pair Ω_{nl} with any S tangent tensor F with $\|F\|_{L^2(S)}\leq 1$. By (6.58),

$$\begin{aligned}\langle\Omega_{nl},F\rangle &= \langle P_l(an\check{\rho}P_n^2 G), P_l{}^\star\mathcal{D}^{-1}F\rangle\\ &\lesssim 2^{\alpha n}r^{-\alpha_0}\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\|P_n G\|_{L_x^2}.\end{aligned}$$

Hence, by (6.32) in Lemma 6.3,

$$\|\Omega_{nl}\|_{L_t^b L_x^2} \lesssim 2^{\alpha n-(\frac{1}{2}+\frac{1}{b})n} r^{-\alpha_0+\frac{1}{2}+\frac{1}{b}} \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2} \mathcal{N}_1(G).$$

Consequently,

$$\sum_{0<l<n} \|\Omega_{nl}\|_{L_t^b L_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2} \mathcal{N}_1(G).$$

(6.56) follows. □

We are ready to prove Proposition 6.9.

Proof of Proposition 6.9. (6.50) can be derived by using (6.49), Theorem 4.3 and (6.8).

Now we consider (6.49). By letting $F = 1$ in (6.46), we obtain for $4 < b < \infty$ that

$$\|r^{-\frac{1}{b}} \mathcal{D}_2^{-1} Err\|_{L_t^b L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Thus we only need to consider $\mathcal{D}_1^{-1}\widetilde{Err}$.

By definition of $\widetilde{Err}$ in (6.6), in view of (2.6), we rewrite $\widetilde{Err}$ symbolically as follows

$$\widetilde{Err} = {}^\star\mathcal{D}_1^{-1}(an\mathrm{tr}\chi\underline{\beta} + an\underline{A}\cdot(\nabla\!\!\!/ A + A\cdot A + r^{-1}A)) + {}^\star\mathcal{D}_1^{-1}(an(\zeta\cdot\rho - \zeta^\star\sigma)). \tag{6.61}$$

By Propositions 4.3 and 6.5,

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-2}(an\underline{A}\cdot(\nabla\!\!\!/ A + r^{-1}A + A\cdot A))\|_{L_t^b L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

According to (6.61), we consider $\mathcal{W} = \|r^{-\frac{1}{b}}\mathcal{D}^{-2}(an\mathrm{tr}\chi\underline{\beta})\|_{L_t^b L_x^2}$, and

$$\mathcal{U} = \|r^{-\frac{1}{b}}\mathcal{D}^{-2}(an\underline{\zeta}\cdot\check{\rho})\|_{L_t^b L_x^2}, \quad \mathcal{V} = \|r^{-\frac{1}{b}}\mathcal{D}^{-2}(an\underline{\zeta}^\star\check{\sigma})\|_{L_t^b L_x^2}.$$

By $\underline{\beta} = {}^\star\mathcal{D}_1{}^\star\mathcal{D}_1^{-1}\underline{\beta}$, also using Propositions 3.4 and 4.3, (2.28), (6.17) and (6.22)

$$\begin{aligned}
\mathcal{W} &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-2}\left(({}^\star\mathcal{D}_1(an\mathrm{tr}\chi{}^\star\mathcal{D}_1^{-1}\underline{\beta})) - \nabla\!\!\!/(an\mathrm{tr}\chi){}^\star\mathcal{D}_1^{-1}\underline{\beta}\right)\|_{L_t^b L_x^2} \\
&\lesssim \|r^{1-\frac{1}{b}}(an\mathrm{tr}\chi{}^\star\mathcal{D}_1^{-1}\underline{\beta})\|_{L_t^b L_x^2} + \|r^{\frac{3}{2}-\frac{1}{b}}\nabla\!\!\!/(an\mathrm{tr}\chi){}^\star\mathcal{D}_1^{-1}\underline{\beta}\|_{L_t^b L_x^{4/3}} \\
&\lesssim \|r^{-\frac{1}{b}}{}^\star\mathcal{D}_1^{-1}\underline{\beta}\|_{L_t^b L_x^2} + \|r^{1-\frac{1}{b}}{}^\star\mathcal{D}_1^{-1}\underline{\beta}\|_{L_t^b L_x^4}\|r^{\frac{1}{2}}\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L_t^\infty L_x^2} \\
&\lesssim \mathcal{N}_1(r^{1/2}{}^\star\mathcal{D}_1^{-1}\underline{\beta})(\|r^{\frac{1}{2}}\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L_t^\infty L_x^2} + 1) \lesssim \Delta_0^2 + \mathcal{R}_0.
\end{aligned}$$

By (2.7), clearly $\check{\sigma} = curl\zeta$. Thus by Propositions 4.3 and 6.5, we obtain

$$\mathcal{V} = \|r^{-\frac{1}{b}}\mathcal{D}^{-1}\mathcal{E}_2(\zeta)\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(\zeta)\mathcal{N}_1(\underline{\zeta}) \lesssim (\Delta_0^2 + \mathcal{R}_0)^2.$$

By Proposition 3.4 and (6.56), we have

$$\begin{aligned}
\mathcal{U} &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-2}(an\check{\rho}\cdot\underline{\zeta})\|_{L_t^b L_x^2} \lesssim \|r^{-\frac{1}{b}+1}\mathcal{D}^{-1}(an\check{\rho}\cdot\underline{\zeta})\|_{L_t^b L_x^2} \\
&\lesssim \mathcal{N}_1(\underline{\zeta})\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)^2
\end{aligned}$$

where we employed (2.79) and (3.4) to obtain the last inequality. □

Case 2: $\mathcal{F} = (a\delta + 2a\lambda)$**.** We first give a slightly stronger result than Proposition 6.8 for Case 2.

Proposition 6.10. *There holds for* $4 < b < \infty$*,*

$$\|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}_t\mathcal{D}_1^{-1}(a\delta + 2a\lambda)\|_{L_t^b L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{6.62}$$

Proof. It is easy to see

$$\|r^{-\frac12-\frac1b}\mathcal{D}_t\mathcal{D}_1^{-1}(a\delta+2a\lambda)\|_{L_t^bL_x^2}\lesssim\|r^{-\frac12-\frac1b}\mathcal{D}_1^{-1}\mathcal{D}_t(a\delta+2a\lambda)\|_{L_t^bL_x^2}$$
$$+\|r^{-\frac12-\frac1b}[\mathcal{D}_t,\mathcal{D}_1^{-1}](a\delta+2a\lambda)\|_{L_t^bL_x^2}. \tag{6.63}$$

By (2.16) and (2.7),

$$an\underline{\zeta}=\mathcal{D}_1^{-1}\mathcal{D}_t(a\delta+2a\lambda)-\mathcal{D}_1^{-1}(an(\check\rho,\check\sigma))+\mathcal{D}_1^{-1}\mathrm{err}_1 \tag{6.64}$$

where, symbolically, $\mathrm{err}_1:=an(a\not{\pi}\mathrm{tr}\chi+A\cdot A)$.

$$\|r^{-\frac12-\frac1b}\mathcal{D}_1^{-1}\mathcal{D}_t(a\delta+2a\lambda)\|_{L_t^bL_x^2}$$
$$\lesssim\|r^{-\frac12-\frac1b}(an\underline{\zeta})\|_{L_t^bL_x^2}+\|r^{-\frac12-\frac1b}\mathcal{D}_1^{-1}(an(\check\rho,\check\sigma))\|_{L_t^bL_x^2} \tag{6.65}$$
$$+\|r^{-\frac12-\frac1b}\mathcal{D}_1^{-1}\mathrm{err}_1\|_{L_t^bL_x^2}. \tag{6.66}$$

By (2.28) and Proposition 6.6, the two terms on the right of (6.65) can be bounded by

$$\mathcal{N}_1(\underline{\zeta})+\Delta_0^2+\mathcal{R}_0\lesssim\Delta_0^2+\mathcal{R}_0.$$

For (6.66), by Proposition 3.4 and (6.17), also in view of (2.23) and (2.28), we deduce

$$\begin{aligned}\|r^{-\frac12-\frac1b}\mathcal{D}_1^{-1}\mathrm{err}_1\|_{L_t^bL_x^2}&\lesssim\|r^{\frac12-\frac1b}\mathrm{err}_1\|_{L_t^bL_x^2}\\&\lesssim\|r^{\frac12-\frac1b}a\not{\pi}\mathrm{tr}\chi\|_{L_t^bL_x^2}+\|r^{\frac12-\frac1b}A\cdot A\|_{L_t^bL_x^2}\\&\lesssim\|r^{-\frac12-\frac1b}\not{\pi}\|_{L_t^bL_x^2}+\|A\cdot A\|_{L_t^\infty L_x^2}\\&\lesssim\mathcal{N}_1(\not{\pi})+\|A\|^2_{L_t^\infty L_x^4}\lesssim\Delta_0^2+\mathcal{R}_0.\end{aligned} \tag{6.67}$$

Thus, we proved

$$\|r^{-\frac12-\frac1b}\mathcal{D}_1^{-1}\mathcal{D}_t(a\delta+2a\lambda)\|_{L_t^bL_x^2}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{6.68}$$

Repeat the derivation for (6.36), also using Lemma 3.1,

$$\begin{aligned}\|r^{-\frac12-\frac1b}[\mathcal{D}_1^{-1},\mathcal{D}_t](a\delta+2a\lambda)\|_{L_t^bL_x^2}&\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(r^{-1}\mathcal{D}_1^{-1}(a\delta+2a\lambda))\\&\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(a\not{\pi})\lesssim(\Delta_0^2+\mathcal{R}_0)^2.\end{aligned}$$

Hence (6.62) is proved. □

Lemma 6.6.

$$\|err_1\|_{\mathcal{P}^0}+\|\nabla\!\!\!\!/\,\mathcal{D}^{-1}err_1\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0, \tag{6.69}$$

Proof. By (4.21), we have

$$\|an\cdot A\cdot A\|_{\mathcal{P}^0}\lesssim\Delta_0\mathcal{N}_1(A)\lesssim\Delta_0^2+\mathcal{R}_0 \tag{6.70}$$

Using (4.21) and (4.13), we obtain

$$\|a^2n\mathrm{tr}\chi\not{\pi}\|_{\mathcal{P}^0}\lesssim\mathcal{N}_1(a\not{\pi})\lesssim\Delta_0^2+\mathcal{R}_0. \tag{6.71}$$

Thus the first inequality of (6.69) is proved. The second one can be proved by combining (6.70), (6.71), (6.67) and Theorem 4.3. □

6.7. **Decomposition for commutators.** In order to prove Proposition 6.1, it remains to decompose the "bad" terms which have not been treated in Proposition 6.2, i.e

$$an\beta\cdot\mathcal{D}^{-1}\mathcal{F},\ \nabla\!\!\!\!/\,\mathcal{D}^{-1}(an\beta\cdot\mathcal{D}^{-1}\mathcal{F}),$$

with $\mathcal{F}$ either $\mathcal{D}^{-1}\check{R}$ or $(a\delta+2a\lambda)$. In view of (6.23) and Proposition 6.8, the assumptions in the following theorem are satisfied with $F=\mathcal{D}^{-1}\mathcal{F}$. Then the proof of Proposition 6.1 is complete by using the following

Theorem 6.1. *Assume that F is an S-tangent tensor field of appropriate order on $\mathcal{H}$ verifying $\mathcal{N}_2(F)<\infty$ and $\|r^{-\frac{1}{b}}\nabla\!\!\!\!/_L F\|_{L_t^b L_x^2}<\infty$ with $4<b<\infty$. Then we have*

(i) *There exists a 1-form E_0 such that*[11]

$$an\beta=\mathcal{D}_t\mathcal{D}^{-1}\check{R}+E_0\quad \textit{with}\quad \|E_0\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0 \tag{6.72}$$

(ii) *There exists a decomposition $an\beta\cdot F=\mathcal{D}_tP+E$, where P and E are tensor fields of the same type as $an\beta\cdot F$ with*

$$\lim_{t\to 0}\|P\|_{L_x^\infty}=0 \tag{6.73}$$

and the estimates

$$\mathcal{N}_1(P)\lesssim\Delta_0\mathcal{N}_2(F),\ \|E\|_{\mathcal{P}^0}\lesssim\Delta_0\cdot\left(\mathcal{N}_2(F)+\|r^{-\frac{1}{b}}\nabla\!\!\!\!/_L F\|_{L_t^bL_x^2}\right). \tag{6.74}$$

(iii) *There exist tensors $\bar{P}$ and $\bar{E}$ verifying (6.74) so that*

$$\nabla\!\!\!\!/\,\mathcal{D}^{-1}(an\beta\cdot F)=\mathcal{D}_t\bar{P}+\bar{E}, \tag{6.75}$$

where $\mathcal{D}$ denote either $\mathcal{D}_1$ or $\mathcal{D}_2$, and

$$\lim_{t\to 0}\|\bar{P}\|_{L_x^\infty}<\infty \tag{6.76}$$

Proof. In view of (6.6), we have

$$an\beta=\mathcal{D}_t\mathcal{D}^{-1}\check{R}+C_0(\check{R})+Err. \tag{6.77}$$

This proves (i) by noting that $E_0:=Err+C_0(\check{R})$ satisfies $\|E_0\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0$ in view of (6.8) and (6.11).

Now we prove (ii). We have from (6.77) that

$$an\beta\cdot F=(\mathcal{D}_t\mathcal{D}_1^{-1}\check{R}+Err+C_0(\check{R}))\cdot F=\mathcal{D}_t(\mathcal{D}_1^{-1}\check{R}\cdot F)+E_1^B+E_1^G,$$

where

$$E_1^B:=-\mathcal{D}_1^{-1}\check{R}\cdot\mathcal{D}_tF\quad\text{and}\quad E_1^G:=(Err+C_0(\check{R}))\cdot F.$$

By (4.18), (6.11) and (6.8) we obtain

$$\|E_1^G\|_{\mathcal{P}^0}\lesssim\mathcal{N}_2(F)\left(\|Err\|_{\mathcal{P}^0}+\|C_0(\check{R})\|_{\mathcal{P}^0}\right)\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F). \tag{6.78}$$

By (4.17) and (6.22) we have

$$\begin{aligned}\|E_1^B\|_{\mathcal{P}^0}&\lesssim\mathcal{N}_1(\mathcal{D}^{-1}\check{R})(\|r^{-\frac{1}{b}}\nabla\!\!\!\!/_LF\|_{L_t^bL_x^2}+\|r^{\frac{1}{2}}\nabla\!\!\!\!/\,\mathcal{D}_tF\|_{L_t^2L_x^2})\\&\lesssim(\mathcal{R}_0+\Delta_0^2)(\mathcal{N}_2(F)+\|r^{-\frac{1}{b}}\nabla\!\!\!\!/_LF\|_{L_t^bL_x^2}).\end{aligned}$$

Now we set

$$P_1:=\mathcal{D}_1^{-1}\check{R}\cdot F\quad\text{and}\quad E_1:=E_1^B+E_1^G, \tag{6.79}$$

[11] In Theorem 6.1 and the following proofs, $\check{R}=(\check{\rho},-\check{\sigma})$ and $C_0(\check{R})=[\mathcal{D}_t,\mathcal{D}_1^{-1}](\check{\rho},-\check{\sigma})$, since the other case that $\check{R}=\underline{\beta}$ will not come up here.

from the above estimates we have

$$\|E_1\|_{\mathcal{P}^0} \lesssim (\Delta_0^2+\mathcal{R}_0)(\mathcal{N}_2(F)+\|r^{-\frac{1}{b}}\nabla\!\!\!/_L F\|_{L_t^b L_x^2}).$$

In order to estimate $\mathcal{N}_1(P_1)$, let us estimate $\|E_1\|_{L^2}$ first. By using Hölder's inequality and (**Sob**), we can obtain

$$\begin{aligned}\|E_1^B\|_{L^2} &= \|\mathcal{D}^{-1}\check{R}\cdot\nabla\!\!\!/_L F\|_{L^2} \lesssim \|\mathcal{D}^{-1}\check{R}\|_{L_t^\infty L_x^4}\|\nabla\!\!\!/_L F\|_{L_t^2 L_x^4}\\ &\lesssim \mathcal{N}_1(\mathcal{D}^{-1}\check{R})(\|\nabla\!\!\!/\mathcal{D}_t F\|_{L^2}+\|r^{-\frac{1}{2}}\nabla\!\!\!/_L F\|_{L^2}),\end{aligned}$$

and by using $\|E_1^G\|_{L^2(\mathcal{H})}\lesssim\|E_1^G\|_{\mathcal{P}^0}$ and (6.78) we can obtain

$$\|E_1^G\|_{L^2}\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F).$$

Therefore

$$\|E_1\|_{L^2}\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F). \tag{6.80}$$

Now we show

$$\mathcal{N}_1(P_1)\lesssim\mathcal{N}_2(F)(\Delta_0^2+\mathcal{R}_0). \tag{6.81}$$

With the help of $\mathcal{D}_t P_1 = an\beta\cdot F - E_1$ and (6.80) we can estimate $\|\nabla\!\!\!/_L P_1\|_{L^2}$ as follows

$$\|\nabla\!\!\!/_L P_1\|_{L^2}\lesssim\|\beta\cdot F\|_{L_t^2L_x^2}+\|E_1\|_{L^2}\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F).$$

Similar to [2, Section 6.12], we get $\|\nabla P_1\|_{L_t^2L_x^2}\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F)$. We complete the proof of (6.74).

By (6.79)

$$\|P_1\|_{L_x^\infty}\le\|F\|_{L_x^\infty}\|\mathcal{D}_1^{-1}\check{R}\|_{L_x^\infty}\lesssim r^{\frac{1}{2}}\|\mathcal{D}_1^{-1}\check{R}\|_{L_x^\infty}\mathcal{N}_2(F) \tag{6.82}$$

Since $\mathcal{N}_2(F)<\infty$ and $\lim_{t\to0}\|\mathcal{D}_1^{-1}\check{R}\|_{L_x^\infty}<\infty$, (6.73) follows by letting $t\to0$ in (6.82). Therefore (ii) is proved.

Finally we prove (iii) by using the iteration procedure in [2, Section 6.12]. (6.76) will be proved in Section 8. Let $P_0:=\mathcal{D}F$, then we can apply (ii) to construct recurrently two sequences of S-tangent tensor fields $\{P_i\}$ and $\{E_i\}$ such that

$$an\beta\cdot\mathcal{D}^{-1}P_{i-1}=\mathcal{D}_tP_i+E_i \tag{6.83}$$

where $\mathcal{D}^{-1}$ denote either $\mathcal{D}_1^{-1}$ or $\mathcal{D}_2^{-1}$ and

$$\mathcal{N}_1(P_i)\le C(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(\mathcal{D}^{-1}P_{i-1}), \tag{6.84}$$

$$\|E_i\|_{\mathcal{P}^0}\le C(\Delta_0^2+\mathcal{R}_0)\left(\mathcal{N}_2(\mathcal{D}^{-1}P_{i-1})+\|r^{-\frac{1}{b}}\nabla\!\!\!/_L\mathcal{D}^{-1}P_{i-1}\|_{L_t^bL_x^2}\right). \tag{6.85}$$

Such P_i and $E_i=E_i^B+E_i^G$ can be constructed as in the proof of (ii). Then for $i=1,2\cdots$,

$$P_i=\mathcal{D}_1^{-1}\check{R}\cdot\mathcal{D}^{-1}P_{i-1},\qquad P_0=\mathcal{D}F \tag{6.86}$$

$$E_i^B:=-\mathcal{D}_1^{-1}\check{R}\cdot\mathcal{D}_t\mathcal{D}^{-1}P_{i-1},\qquad E_i^G:=(Err+C_0(\check{R}))\cdot\mathcal{D}^{-1}P_{i-1}. \tag{6.87}$$

In particular, P_1 and E_1 have been given by (6.79).

With the above definition of P_k and E_k, using (6.83)

$$\nabla\!\!\!/\mathcal{D}^{-1}(an\beta\cdot F)=\mathcal{D}_t\bar{P}_k+\nabla\!\!\!/\mathcal{D}^{-1}(\mathcal{D}_tP_k)+\bar{E}_k,$$

where

$$\bar{P}_k = \nabla\!\!\!/ \mathcal{D}^{-1}(P_1 + ... + P_{k-1}) + P_2 + ... + P_k \tag{6.88}$$
$$\bar{E}_k = [\nabla\!\!\!/ \mathcal{D}^{-1}, \mathcal{D}_t]_g(P_1 + ... + P_{k-1}) + \nabla\!\!\!/ \mathcal{D}^{-1}(E_1 + ... + E_k) + E_2 + ... + E_k.$$

By using Lemma 3.1, it is easy to see from (6.84) that

$$\mathcal{N}_1(P_k) \le (C(\Delta_0^2 + \mathcal{R}_0))^k \mathcal{N}_2(F). \tag{6.89}$$

Moreover we have

Proposition 6.11. *For $\{P_k\}_{k=1}^\infty$ and $\{E_k\}_{k=1}^\infty$ there hold*

$$\|r^{-\frac{1}{b}} \nabla\!\!\!/_L \mathcal{D}^{-1} P_k\|_{L_t^b L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)(\mathcal{N}_2(\mathcal{D}^{-1}P_{k-1}) + \|\nabla\!\!\!/_L \mathcal{D}^{-1} P_{k-1}\|_{L_t^b L_x^2}), \tag{6.90}$$

$$\|\nabla\!\!\!/ \mathcal{D}^{-1} E_k\|_{\mathcal{P}^0} \lesssim \|E_k\|_{\mathcal{P}^0} + (\Delta_0^2 + \mathcal{R}_0)(\mathcal{N}_2(\mathcal{D}^{-1}P_{k-1}) + \|\nabla\!\!\!/_L \mathcal{D}^{-1} P_{k-1}\|_{L_t^b L_x^2}). \tag{6.91}$$

We will prove this result at the end of this section. We observe that Lemma 3.1, (6.90), (6.84) and (6.85) clearly imply

$$\|E_k\|_{\mathcal{P}^0} \le (C(\Delta_0^2 + \mathcal{R}_0))^k \left(\mathcal{N}_2(F) + \|r^{-\frac{1}{b}} \nabla\!\!\!/_L F\|_{L_t^b L_x^2}\right). \tag{6.92}$$

It follows from (6.89), (6.91), (6.92) and (6.26) that

$$\mathcal{N}_1(\bar{P}_k - \bar{P}_j) \le \mathcal{N}_2(F) \sum_{j \le m \le k-1} (C(\Delta_0^2 + \mathcal{R}_0))^m \lesssim (C(\Delta_0^2 + \mathcal{R}_0))^j \mathcal{N}_2(F),$$

and

$$\|\bar{E}_k - \bar{E}_j\|_{\mathcal{P}^0} \le \big(\mathcal{N}_2(F) + \|r^{-\frac{1}{b}} \nabla\!\!\!/_L F\|_{L_t^b L_x^2}\big) \sum_{j \le m \le k-1} (C(\Delta_0^2 + \mathcal{R}_0))^m$$
$$\lesssim (C(\Delta_0^2 + \mathcal{R}_0))^j \big(\mathcal{N}_2(F) + \|r^{-\frac{1}{b}} \nabla\!\!\!/_L F\|_{L_t^b L_x^2}\big).$$

Therefore $\{\bar{P}_k\}$ forms a Cauchy sequence relative to the norm $\mathcal{N}_1(\cdot)$, while $\{\bar{E}_k\}$ forms a Cauchy sequence relative to the $\mathcal{P}^0$ norm. Denote by $\bar{P}$ and $\bar{E}$ their corresponding limits, we have

$$\mathcal{N}_1(\bar{P}) \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(F) \quad \text{and} \quad \|\bar{E}\|_{\mathcal{P}^0} \lesssim (\Delta_0^2 + \mathcal{R}_0)\big(\mathcal{N}_2(F) + \|r^{-\frac{1}{b}} \nabla\!\!\!/_L F\|_{L_t^b L_x^2}\big).$$

We also observe that for sufficiently small Δ_0,

$$\|\nabla\!\!\!/ \mathcal{D}^{-1}(an\beta \cdot F) - \mathcal{D}_t \bar{P}_k - \bar{E}_k\|_{L^2} = \|\nabla\!\!\!/ \mathcal{D}^{-1}(\mathcal{D}_t P_k)\|_{L^2} \lesssim \mathcal{N}_1(P_k).$$

Letting $k \to +\infty$, we get

$$\|\nabla\!\!\!/ \mathcal{D}^{-1}(an\beta \cdot F) - \mathcal{D}_t \bar{P} - \bar{E}\|_{L_t^2 L_x^2} = 0.$$

Hence $\nabla\!\!\!/ \mathcal{D}^{-1}(an\beta \cdot F) = \mathcal{D}_t \bar{P} + \bar{E}$. This completes the proof of (6.75) in (iii). (6.76) will be proved in Appendix.

Now we conclude this section by proving Proposition 6.11. We first prove (6.91). By using (4.16) we have

$$\|\nabla\!\!\!/ \mathcal{D}^{-1} E_k\|_{\mathcal{P}^0} \lesssim \|E_k\|_{\mathcal{P}^0} + (\Delta_0^2 + \mathcal{R}_0)\|\mathcal{D}^{-1} E_k\|_{L_t^b L_x^2}^q \|E_k\|_{L^2}^{1-q},$$

where $4 < b < \infty$ and $1/2 < q < 1$.

Thus it suffices to show for $4 < b < \infty$ that

$$\|r^{-\frac{1}{b}} \mathcal{D}^{-1} E_k\|_{L_t^b L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0) \left(\mathcal{N}_2(\mathcal{D}^{-1}P_{k-1}) + \|\nabla\!\!\!/_L \mathcal{D}^{-1} P_{k-1}\|_{L_t^b L_x^2}\right). \tag{6.93}$$

By the construction of P_k and E_k, it suffices to show it for $k=1$. To this end, in view of $E_1=E_1^G+E_1^B$, we can complete the proof by using Proposition 6.7 for $\|r^{-\frac{1}{b}}\mathcal{D}^{-1}E_1^G\|_{L_t^bL_x^2}$ and the estimate

$$\begin{aligned}\|r^{-\frac{1}{b}}\mathcal{D}^{-1}E_1^B\|_{L_t^bL_x^2} &\lesssim \|r^{\frac{1}{2}-\frac{1}{b}}E_1^B\|_{L_t^bL_x^{4/3}} \lesssim \|\mathcal{D}_1^{-1}\check{R}\|_{L_t^\infty L_x^4}\|\mathcal{D}_tF\|_{L_t^bL_x^2}\\ &\lesssim \mathcal{N}_1(\mathcal{D}^{-1}\check{R})\|\mathcal{D}_tF\|_{L_t^bL_x^2} \lesssim (\Delta_0^2+\mathcal{R}_0)\|\nabla\!\!\!\!/_L F\|_{L_t^bL_x^2}.\end{aligned}$$

which follows from Proposition 4.3, Hölder inequality, (**SobM1**) and (6.22).

In order to prove (6.90), we first note that

$$\|r^{-\frac{1}{b}}\nabla\!\!\!\!/_L\mathcal{D}^{-1}P_k\|_{L_t^bL_x^2} \lesssim \|r^{-\frac{1}{b}}[\mathcal{D}_t,\mathcal{D}^{-1}]P_k\|_{L_t^bL_x^2}+\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\mathcal{D}_tP_k\|_{L_t^bL_x^2}. \tag{6.94}$$

By using (6.36), the first term on the right hand side of (6.94) can be estimated as

$$\|r^{-\frac{1}{b}}C_0(P_k)\|_{L_t^bL_x^2} \lesssim \mathcal{N}_1(r^{-\frac{1}{2}}\mathcal{D}^{-1}P_k) \lesssim \mathcal{N}_2(r^{\frac{1}{2}}\mathcal{D}^{-1}P_k) \lesssim \mathcal{N}_1(P_k),$$

while by using (6.83), (6.93) and (6.35), the second term can be estimated as

$$\begin{aligned}\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\mathcal{D}_tP_k\|_{L_t^bL_x^2} &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(an\beta\cdot\mathcal{D}^{-1}P_{k-1}-E_k)\|_{L_t^bL_x^2}\\ &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(an\beta\cdot\mathcal{D}^{-1}P_{k-1})\|_{L_t^bL_x^2}+\|r^{-\frac{1}{b}}\mathcal{D}^{-1}E_k\|_{L_t^bL_x^2}\\ &\lesssim (\Delta_0^2+\mathcal{R}_0)(\mathcal{N}_2(\mathcal{D}^{-1}P_{k-1})+\|\nabla\!\!\!\!/_L\mathcal{D}^{-1}P_{k-1}\|_{L_t^bL_x^2}).\end{aligned}$$

Therefore (6.90) is proved. □

7. Main estimates

Lemma 7.1. *Let $F=\nabla\!\!\!\!/ \mathrm{tr}\chi,\mu, A\cdot\underline{A}, r^{-1}\underline{A}$, there holds*

$$\|\nabla\!\!\!\!/\mathcal{D}^{-1}(anF)\|_{\mathcal{P}^0} \lesssim \Delta_0^2+\mathcal{R}_0+\|F\|_{\mathcal{P}^0} \tag{7.1}$$

where $\mathcal{D}^{-1}$ is one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$, ${}^\star\mathcal{D}_1^{-1}$.

Proof. By Proposition 4.3 and (2.79), we have

$$\|\mathcal{D}^{-1}(anF)\|_{L_t^bL_x^2} \lesssim \|anrF\|_{L_t^bL_x^2} \lesssim \|rF\|_{L_t^bL_x^2} \lesssim \Delta_0^2+\mathcal{R}_0.$$

By (2.79), $\|F\|_{L^2(\mathcal{H})}\lesssim\Delta_0^2+\mathcal{R}_0$. We can infer from Theorem 4.3 and (4.13) that

$$\|\nabla\!\!\!\!/\mathcal{D}^{-1}(anF)\|_{\mathcal{P}^0} \lesssim \|anF\|_{\mathcal{P}^0}+\Delta_0^2+\mathcal{R}_0 \lesssim \|F\|_{\mathcal{P}^0}+\Delta_0^2+\mathcal{R}_0.$$

as desired. □

Now we improve BA1 with the help of Theorem 5.1.

7.1. Estimates for ν and $|a-1|$.

Proposition 7.1.

$$\|\nu\|_{L_\omega^\infty L_t^2} \lesssim \Delta_0^2+\mathcal{R}_0, \quad |a-1|\le\frac{1}{4}.$$

Proof. We rewrite (2.17) as follows

$$-\nabla\!\!\!\!/\nu=\nabla\!\!\!\!/_L(\nabla\!\!\!\!/ a)+\mathrm{err}_2, \text{ and } \mathrm{err}_2=\frac{1}{2}\mathrm{tr}\chi\nabla\!\!\!\!/ a+\hat{\chi}\cdot\nabla\!\!\!\!/ a+A\cdot\nu.$$

Let us denote symbolically $\mathrm{err}_2=\mathrm{tr}\chi\cdot\underline{A}+A\cdot\underline{A}$, hence

$$-\nabla\!\!\!\!/(an\nu)=\mathcal{D}_t\nabla\!\!\!\!/ a+an\widetilde{\mathrm{err}}_2$$

with $\widetilde{\mathrm{err}}_2=-\nabla\!\!\!\!/\log(an)\nu+\mathrm{err}_2=A\cdot\underline{A}+r^{-1}\underline{A}$. By (4.21) we can obtain

$$\|an\widetilde{\mathrm{err}}_2\|_{\mathcal{P}^0} \lesssim \Delta_0^2+\mathcal{R}_0.$$

Applying Theorem 5.1 to $P = \nabla\!\!\!/ a$ and $E = an \cdot \widetilde{\mathrm{err}}_2$, we have

$$\|\nu\|_{L_\omega^\infty L_t^2} \lesssim \mathcal{N}_1(\nu) + \mathcal{N}_1(P) + \|an\widetilde{\mathrm{err}}_2\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0,$$

where in view of (2.79), $\mathcal{N}_1(\nu) + \mathcal{N}_1(P) \lesssim \Delta_0^2 + \mathcal{R}_0$.

In view of $\nu := -\frac{d}{ds}a$ and $a(p) = 1$,

$$|a-1| \le \int_0^t |\nu| nadt' \lesssim \|\nu\|_{L_\omega^\infty L_t^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

With $\Delta_0^2 + \mathcal{R}_0$ being sufficiently small, $|a-1| \le \frac{1}{4}$ can be achieved. Then Proposition 7.1 follows. $\square$

7.2. **Estimate for $\underline{\zeta}$.**

Proposition 7.2.

$$\|\underline{\zeta}\|_{L_\omega^\infty L_t^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Proof. By (6.6)

$$\begin{aligned}\nabla\!\!\!/ \mathcal{D}_1^{-1}\left(an(\rho,\sigma)\right) &= \nabla\!\!\!/ \mathcal{D}_1^{-1\star}\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta} - \nabla\!\!\!/ \mathcal{D}_1^{-1}\widetilde{E}rr \\ &= \mathcal{D}_t\nabla\!\!\!/ \mathcal{D}^{-2}\check{R} + C(\check{R}) + \nabla\!\!\!/ \mathcal{D}^{-1}\mathfrak{F}.\end{aligned}$$

By Proposition 6.1, there exists P and E such that $C(\check{R}) = \mathcal{D}_t P + E$.

Let $\tilde{P} = P + \nabla\!\!\!/ \mathcal{D}^{-2}\check{R}$ and $\tilde{E} = \nabla\!\!\!/ \mathcal{D}^{-1}\mathfrak{F} + E$. Then by (6.23) and (6.50)

$$\nabla\!\!\!/ \mathcal{D}_1^{-1}\left(an(\rho,\sigma)\right) = \mathcal{D}_t\tilde{P} + \tilde{E}, \quad \mathcal{N}_1(\tilde{P}) + \|\tilde{E}\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

In view of (6.64),

$$\begin{aligned}\nabla\!\!\!/ (an\underline{\zeta}) &= \nabla\!\!\!/ \mathcal{D}_1^{-1}(\mathcal{D}_t(a\delta + 2a\lambda)) + \nabla\!\!\!/ \mathcal{D}_1^{-1}(an(\rho,\sigma)) + \nabla\!\!\!/ \mathcal{D}_1^{-1}\mathrm{err}_1, \\ &= \mathcal{D}_t\nabla\!\!\!/ \mathcal{D}_1^{-1}(a\delta + 2a\lambda) + [\nabla\!\!\!/ \mathcal{D}_1^{-1}, \mathcal{D}_t](a\delta + 2a\lambda) + \nabla\!\!\!/ \mathcal{D}_1^{-1}\mathrm{err}_1 + \mathcal{D}_t\tilde{P} + \tilde{E}.\end{aligned} \tag{7.2}$$

In view of Proposition 6.1, there exists P' an E' such that

$$[\nabla\!\!\!/ \mathcal{D}_1^{-1}, \mathcal{D}_t](a\delta + 2a\lambda) = \mathcal{D}_t P' + E', \text{ with } \mathcal{N}_1(P') + \|E'\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Let $P'' = P' + \nabla\!\!\!/ \mathcal{D}_1^{-1}(a\delta + 2a\lambda)$, we conclude that

$$\mathcal{D}_t\nabla\!\!\!/ \mathcal{D}_1^{-1}(a\delta + 2a\lambda) + [\nabla\!\!\!/ \mathcal{D}_1^{-1}, \mathcal{D}_t](a\delta + 2a\lambda) = \mathcal{D}_t P'' + E'.$$

By (6.23), $\mathcal{N}_1(P'') \lesssim \Delta_0^2 + \mathcal{R}_0$. Hence, we obtain

$$\nabla\!\!\!/ (an\underline{\zeta}) = \mathcal{D}_t P_3 + E_3,$$

where $E_3 = E' + \nabla\!\!\!/ \mathcal{D}_1^{-1}\mathrm{err}_1 + \tilde{E}$ and $P_3 = P'' + \tilde{P}$. Also using Lemma 6.6 for $\|\nabla\!\!\!/ \mathcal{D}_1^{-1}\mathrm{err}_1\|_{\mathcal{P}^0}$, we can conclude that

$$\|E_3\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \qquad \mathcal{N}_1(P_3) \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Proposition 7.2 then follows by using Theorem 5.1 and $\mathcal{N}_1(\underline{\zeta}) \lesssim \Delta_0^2 + \mathcal{R}_0$. $\square$

7.3. **Estimate for $\hat{\chi}$.**

Proposition 7.3.
$$\|\hat{\chi}\|_{L_\omega^\infty L_t^2} \lesssim \Delta_0^2 + \mathcal{R}_0$$

Proof. First, by (2.6),
$$\begin{aligned}\operatorname{div}(an\hat{\chi}) &= \frac{1}{2}an\nabla\!\!\!/\operatorname{tr}\chi + \frac{1}{2}an\operatorname{tr}\chi\zeta - an\beta + an\underline{\zeta}\hat{\chi}\\ &= anM + anA\cdot A + r^{-1}an\zeta - an\beta\end{aligned}$$
from (6.6),
$$an\beta = \mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma}) - \mathfrak{F} \quad \text{with} \quad \mathfrak{F} = Err.$$
Hence
$$\operatorname{div}(an\hat{\chi}) = anM + anA\cdot A + r^{-1}an\zeta - \mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma}) + \mathfrak{F}$$
This gives
$$an\hat{\chi} = -\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma}) + \mathcal{D}_2^{-1}(\mathfrak{F} + anM + anA\cdot A + r^{-1}an\zeta).$$
Set $\mathcal{D}^{-2} = \mathcal{D}_2^{-1}\mathcal{D}_1^{-1}$ and $\mathcal{D}^{-1} = \mathcal{D}_2^{-1}$, we obtain after taking covariant derivatives
$$\nabla\!\!\!/(an\hat{\chi}) = -\nabla\!\!\!/\mathcal{D}^{-2}\mathcal{D}_t\check{R} + F + \nabla\!\!\!/\mathcal{D}^{-1}(anM), \tag{7.3}$$
where $F = \nabla\!\!\!/\mathcal{D}^{-1}(\mathfrak{F} + anA\cdot A + r^{-1}an\zeta)$ and $M = \nabla\!\!\!/\operatorname{tr}\chi$.

By (6.50) and (7.1),
$$\|F\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{7.4}$$

Consider the first term on the right of (7.3). By using the notations in (6.5), we can write
$$\nabla\!\!\!/\mathcal{D}^{-2}\mathcal{D}_t(\check{R}) = \mathcal{D}_t(\nabla\!\!\!/\mathcal{D}^{-2}\check{R}) + C(\check{R}).$$
where, by Proposition 6.1, there exist tensors P' and E' so that $C(\check{R}) = \mathcal{D}_tP' + E'$ and
$$\mathcal{N}_1(P') + \|E'\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \quad \lim_{t\to 0} r\|P'\|_{L_x^\infty} = 0. \tag{7.5}$$
Thus (7.3) becomes
$$\nabla\!\!\!/(an\hat{\chi}) = \mathcal{D}_tP + \nabla\!\!\!/\mathcal{D}^{-1}(anM) + E \tag{7.6}$$
where $P = \nabla\!\!\!/\mathcal{D}^{-2}\check{R} + P'$ and $E = F + E'$. By using (6.23),(7.4) and (7.5)
$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \quad \lim_{t\to 0} r\|P\|_{L_x^\infty} = 0 \tag{7.7}$$
By combining (7.6) with (2.4) we obtain
$$\frac{d}{ds}M + \frac{3}{2}\operatorname{tr}\chi M = -\hat{\chi}\cdot M - 2(an)^{-1}\hat{\chi}(\mathcal{D}_tP + E + \nabla\!\!\!/\mathcal{D}^{-1}(anM)) - \frac{1}{2}(\operatorname{tr}\chi)^2(\zeta + \underline{\zeta}),$$
regarding ι as an element of A, symbolically,
$$\begin{aligned}\nabla\!\!\!/_LM + \frac{3}{2}\operatorname{tr}\chi M = A\cdot M &+ (an)^{-1}\hat{\chi}\cdot(\mathcal{D}_tP + E + \nabla\!\!\!/\mathcal{D}^{-1}(anM))\\ &+ (r^{-1}A + A\cdot A)\cdot A + r^{-2}A.\end{aligned}$$
Then
$$\begin{aligned}\mathcal{D}_t(r^3M) = -\frac{3}{2}r^3an\kappa M &+ r^3an\{A\cdot M + (an)^{-1}\hat{\chi}(\mathcal{D}_tP + E + \nabla\!\!\!/\mathcal{D}^{-1}(anM))\}\\ &+ r^3anA(A\cdot A + r^{-1}A) + ranA.\end{aligned}$$

Let us pair $\nabla\!\!\!/ \mathrm{tr}\chi$ with vector fields X_i in Lemma 4.1, which is still denoted by M. Regarding κ also as an element of A, integrating in t, in view of $\lim_{s\to 0} r\nabla\!\!\!/ \mathrm{tr}\chi = 0$,

$$\begin{aligned} M &= r^{-3}\int_0^t r'^3 anA\cdot M + r'^3\hat{\chi}\cdot\mathcal{D}_t P \\ &+ r^{-3}\int_0^t r'^3 A\cdot (E + \nabla\!\!\!/\mathcal{D}^{-1}(anM) + anA\cdot A + r'^{-1}anA) + anr'Adt'. \end{aligned}$$

Using Lemma 4.1, (5.2) and (5.4) in view of $\lim_{t\to 0}\|\hat{\chi}\|_{L_x^\infty} < \infty$, we can obtain

$$\begin{aligned} &\|r^{-3}\int_0^t r'^3 A\cdot \big(an(M + A\cdot A + r^{-1}A) + E + \nabla\!\!\!/\mathcal{D}^{-1}(anM)\big) + r'^3\hat{\chi}\cdot\mathcal{D}_t P\|_{\mathcal{B}^0} \\ &\lesssim (\mathcal{N}_1(A) + \|A\|_{L_\omega^\infty L_t^2})(\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} + \|\nabla\!\!\!/\mathcal{D}^{-1}(anM)\|_{\mathcal{P}^0} \\ &+ \|anM\|_{\mathcal{P}^0} + \|r^{-1}anA\|_{\mathcal{P}^0} + \|anA\cdot A\|_{\mathcal{P}^0}). \end{aligned}$$

By Proposition 4.2, (2.32), (4.21) and (2.79), we obtain

$$\|r^{-3}\int_0^t anr'A\|_{\mathcal{P}^0} \lesssim \|r^{-1}anA\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(A) \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Hence, in view of (4.13), BA1, (4.21) and (7.1), we can obtain

$$\begin{aligned} \|M\|_{\mathcal{P}^0} &\lesssim \big(\mathcal{N}_1(P) + \|M\|_{\mathcal{P}^0} + \|E\|_{\mathcal{P}^0} + \Delta_0^2 + \mathcal{R}_0\big)\Big(\mathcal{N}_1(A) + \|A\|_{L_\omega^\infty L_t^2}\Big) + \Delta_0^2 + \mathcal{R}_0 \\ &\lesssim \Delta_0\big(\|M\|_{\mathcal{P}^0} + \mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} + \Delta_0^2 + \mathcal{R}_0\big) + \Delta_0^2 + \mathcal{R}_0. \end{aligned} \tag{7.8}$$

Since $0 < \Delta_0 < 1/2$ can be chosen to be sufficiently small such that the first term of (7.8) can be absorbed, in view of (7.7) we then obtain that

$$\|\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{7.9}$$

Thus, by setting $\tilde{E} = E + \nabla\!\!\!/\mathcal{D}^{-1}(anM)$ we obtain from (7.6), (7.1) and (7.9) the decomposition

$$\nabla\!\!\!/(an\hat{\chi}) = \mathcal{D}_t P + \tilde{E} \quad \text{and} \quad \mathcal{N}_1(P) + \|\tilde{E}\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{7.10}$$

By Theorem 5.1 and (2.79), we conclude

$$\|\hat{\chi}\|_{L_\omega^\infty L_t^2} \lesssim \mathcal{N}_1(\hat{\chi}) + \mathcal{N}_1(P) + \|\tilde{E}\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

as expected.

□

Similar to the derivation of (7.9), we can get

$$\|r^{\frac{1}{2}}\nabla\!\!\!/ \mathrm{tr}\chi\|_{\mathcal{B}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{7.11}$$

7.4. **Estimate for ζ.**

Proposition 7.4.

$$\|\zeta\|_{L_\omega^\infty L_t^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Proof. By using (2.8) and (2.9),

$$\begin{aligned} \mathrm{div}(an\zeta) &= an\underline{\zeta}\cdot\zeta - an\mu - an\check{\rho} + \frac{1}{2}a^2 n\delta\mathrm{tr}\chi, \\ \mathrm{curl}\,(an\zeta) &= an\check{\sigma} + an(\zeta + \underline{\zeta})\wedge\zeta. \end{aligned}$$

Symbolically, $\mathcal{D}_1(an\zeta) = anA\cdot A - an(\mu,0) - an(\check\rho,-\check\sigma) + an(a\delta\mathrm{tr}\chi,0)$. Hence

$$an\zeta = -\mathcal{D}_1^{-1}(an(\check\rho,-\check\sigma)) + \mathcal{D}_1^{-1}(anA\cdot\underline{A}) - \mathcal{D}_1^{-1}(an(\mu,0)) + \mathcal{D}_1^{-1}(an(r^{-1}\underline{A},0)).$$

Let J be the involution $(\rho,\sigma)\to(-\rho,\sigma)$, $\mathfrak{F} = \widetilde{Err}$ is given by (6.6),

$$\begin{aligned}\nabla\!\!\!/(an\zeta) =& \nabla\!\!\!/\mathcal{D}_1^{-1}\cdot J\cdot {}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta} + \nabla\!\!\!/\mathcal{D}_1^{-1}\cdot J\cdot\mathfrak{F} - \nabla\!\!\!/\mathcal{D}_1^{-1}(an\mu,0)\\ &+ \nabla\!\!\!/\mathcal{D}_1^{-1}(an\cdot A\cdot A) + \nabla\!\!\!/\mathcal{D}_1^{-1}(r^{-1}an\underline{A},0).\end{aligned}$$

Set $\mathcal{D}^{-2} = \mathcal{D}_1^{-1}\cdot J\cdot{}^\star\mathcal{D}_1^{-1}$ and $\mathcal{D}^{-1} = \mathcal{D}_1^{-1}$. By using (6.5), we get

$$\nabla\!\!\!/(an\zeta) = \mathcal{D}_t\nabla\!\!\!/\mathcal{D}^{-2}\underline{\beta} + C(\check R) + \nabla\!\!\!/\mathcal{D}^{-1}(anM) + F$$

where $M = (\mu,0)$ and $F = \nabla\!\!\!/\mathcal{D}^{-1}\big(\mathfrak{F} + an(A\cdot\underline{A} + r^{-1}\underline{A})\big)$.

By (6.50) and (7.1), we derive $\|F\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0$.

In view of Proposition 6.1, for some tensors $\tilde P$ and $\tilde E$ such that $C(\check R) = \mathcal{D}_t\tilde P + \tilde E$. With $E = \tilde E + F$, $P = \tilde P + \nabla\!\!\!/\mathcal{D}^{-2}\underline{\beta}$, we can write

$$\nabla\!\!\!/(an\zeta) = \mathcal{D}_tP + \nabla\!\!\!/\mathcal{D}^{-1}(anM) + E,$$

and

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \le \Delta_0^2 + \mathcal{R}_0, \qquad \lim_{t\to0} r\|P\|_{L_x^\infty} = 0 \tag{7.12}$$

Let $M = (\mu,0)$, (2.15) can be written as

$$\begin{aligned}\frac{d}{ds}M + \mathrm{tr}\chi M =& 2(an)^{-1}\hat\chi\left(\mathcal{D}_tP + E + \nabla\!\!\!/\mathcal{D}^{-1}(anM)\right) - 2\hat\chi\cdot\underline{\zeta}\cdot\zeta + (\zeta - 2\underline{\zeta})\mathrm{tr}\chi\zeta\\ &+ \nabla\!\!\!/\mathrm{tr}\chi(\zeta-\underline{\zeta}) + \mathrm{tr}\chi\check\rho + \left(\frac14 a^2\mathrm{tr}\chi + \underline{A}\right)|\hat\chi|^2 - \frac12 a\nu(\mathrm{tr}\chi)^2.\end{aligned}$$

Symbolically,

$$\begin{aligned}\frac{d}{ds}M + \mathrm{tr}\chi M =& (an)^{-1}\hat\chi\cdot\big(\mathcal{D}_tP + \nabla\!\!\!/\mathcal{D}^{-1}(anM) + E\big) + \mathrm{tr}\chi\check\rho\\ &+ A\cdot\big(A\cdot\underline{A} + r^{-1}A + \nabla\!\!\!/\mathrm{tr}\chi\big) + r^{-1}a\mathrm{tr}\chi A.\end{aligned}\tag{7.13}$$

In view of $\lim_{t\to0} r^2\mu = 0$ in Lemma 2.1, we deduce

$$M = r^{-2}\int_0^t {r'}^2\left(an\kappa\cdot M + \hat\chi\cdot\mathcal{D}_tP + A\cdot\tilde F + an\mathrm{tr}\chi\check\rho + {r'}^{-1}a^2n\mathrm{tr}\chi A\right)dt' \tag{7.14}$$

with $\tilde F = an(A\cdot\underline{A} + \nabla\!\!\!/\mathrm{tr}\chi + r^{-1}A) + E + \nabla\!\!\!/\mathcal{D}^{-1}(anM)$.

In view of Proposition 5.1, regarding κ as an element of A,

$$\begin{aligned}&\left\|r^{-2}\int_0^t {r'}^2\left(an\kappa\cdot M + \hat\chi\cdot\mathcal{D}_tP + A\cdot\tilde F\right)dt'\right\|_{\mathcal{B}^0}\\ &\qquad\lesssim\left(\mathcal{N}_1(A) + \|A\|_{L_\omega^\infty L_t^2}\right)\left(\|anM\|_{\mathcal{P}^0} + \mathcal{N}_1(P) + \|\tilde F\|_{\mathcal{P}^0}\right).\end{aligned}\tag{7.15}$$

Note that by (4.21), (4.13), (7.1) and (6.17), we deduce

$$\begin{aligned}\|\tilde F\|_{\mathcal{P}^0} &\lesssim \|an(A\cdot\underline{A} + r^{-1}A)\|_{\mathcal{P}^0} + \|an\nabla\!\!\!/\mathrm{tr}\chi\|_{\mathcal{P}^0} + \|E\|_{\mathcal{P}^0} + \|\nabla\!\!\!/\mathcal{D}^{-1}(anM)\|_{\mathcal{P}^0}\\ &\lesssim \|\nabla\!\!\!/\mathrm{tr}\chi\|_{\mathcal{P}^0} + \|E\|_{\mathcal{P}^0} + \|M\|_{\mathcal{P}^0} + \Delta_0^2 + \mathcal{R}_0.\end{aligned}$$

By (7.9) and (7.12)

$$\|\tilde F\|_{\mathcal{P}^0}\lesssim\Delta_0^2 + \mathcal{R}_0 + \|M\|_{\mathcal{P}^0}.$$

Hence, by (6.17), BA1, (4.13) and (7.12)

$$(7.15) \lesssim \Delta_0(\|M\|_{\mathcal{P}^0} + \Delta_0^2 + \mathcal{R}_0).$$

Assuming the following estimate for $\mathcal{P}^0$ norm of the last two terms in (7.14)

$$\left\| r^{-2} \int_0^t r'^2 \left(an\mathrm{tr}\chi\check{\rho} + r'^{-1} a^2 n\mathrm{tr}\chi A \right) dt' \right\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{7.16}$$

since $0 < \Delta_0 < 1/2$ can be chosen sufficiently small, we conclude that

$$\|M\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{7.17}$$

By (7.1) and (2.79), $\|\nabla\!\!\!/ \mathcal{D}^{-1}(anM)\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0$.

In view of (7.12), we have $\nabla\!\!\!/(an\zeta) = \mathcal{D}_t P + E'''$, with $E''' = E + \nabla\!\!\!/ \mathcal{D}^{-1}(anM)$. By Theorem 5.1,

$$\|\zeta\|_{L_\omega^\infty L_t^2} \lesssim \mathcal{N}_1(P) + \|E'''\|_{\mathcal{P}^0} + \mathcal{N}_1(\zeta) \lesssim \Delta_0^2 + \mathcal{R}_0.$$

We now prove (7.16). With the help of (6.4), by letting

$$p' := {}^\star\mathcal{D}_1^{-1}\underline{\beta}, \quad e' = [{}^\star\mathcal{D}_1^{-1}, \mathcal{D}_t]\underline{\beta} - \widetilde{Err} + anA \cdot \underline{A},$$

also noting that (6.11) gives $\|[\mathcal{D}_t, {}^\star\mathcal{D}_1^{-1}]\check{R}\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0$, combined with (6.8), (4.21) and (6.17), we can get the following decomposition

$$an(\check{\rho}, \check{\sigma}) = \mathcal{D}_t p' + e' \quad \text{with} \quad \mathcal{N}_1(p') + \|e'\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{7.18}$$

(7.16) can be derived by establishing the following inequalities

$$\left\| r^{-2} \int_0^t r'^2 \mathrm{tr}\chi \mathcal{D}_t p' dt' \right\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{7.19}$$

$$\left\| r^{-2} \int_0^t r'^2 \mathrm{tr}\chi \left(e' + r'^{-1} anA \right) dt' \right\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{7.20}$$

To prove (7.20), by Proposition 4.2, (2.32), also in view of (4.21) and (7.18), we can obtain

$$\left\| r^{-2} \int_0^t r' \left(e' + r'^{-1} anA \right) dt' \right\|_{\mathcal{P}^0} \lesssim \|e'\|_{\mathcal{P}^0} + \|r^{-1} anA\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

By (5.2) and (7.18), we get

$$\left\| r^{-2} \int_0^t r'^2 \iota \cdot e' dt' \right\|_{\mathcal{B}^0} \lesssim (\mathcal{N}_1(\iota) + \|\iota\|_{L_\omega^\infty L_t^2}) \|e'\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0,$$

and similarly

$$\left\| r^{-2} \int_0^t r' \iota \cdot anA dt' \right\|_{\mathcal{B}^0} \lesssim (\mathcal{N}_1(\iota) + \|\iota\|_{L_\omega^\infty L_t^2}) \|r^{-1} anA\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

The proof of (7.20) is complete.

Now we prove (7.19). Recall that $p' = {}^\star\mathcal{D}_1^{-1}\underline{\beta}$, then

$$\lim_{s \to 0} rp' = 0, \ \|\mathrm{tr}\chi p'\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(p') \lesssim \mathcal{R}_0 + \Delta_0^2. \tag{7.21}$$

Using Proposition 4.2 and (7.21), also in view of (2.1), we derive

$$\begin{aligned}
&\left\|\frac{1}{r^2}\int_0^t {r'}^2 \mathrm{tr}\chi \mathcal{D}_t p' dt'\right\|_{\mathcal{P}^0}\\
&\quad\lesssim \sum_{k>0}\left\|E_k \frac{1}{r}\int_0^t {r'}^2 \mathrm{tr}\chi \mathcal{D}_t p' dt'\right\|_{L_t^2 L_\omega^2} + \left\|\frac{1}{r}\int_0^t {r'}^2 \mathrm{tr}\chi \mathcal{D}_t p' dt'\right\|_{L_t^2 L_\omega^2}\\
&\quad\lesssim \sum_{k>0}\left(\|E_k(r'\mathrm{tr}\chi p')\|_{L_t^2 L_\omega^2} + \|E_k r^{-1}\int_0^t \mathcal{D}_t({r'}^2\mathrm{tr}\chi) p' dt'\|_{L_t^2 L_\omega^2}\right) + \|\nabla\!\!\!\!/_L p'\|_{L_t^2 L_x^2}\\
&\quad\lesssim \|\mathrm{tr}\chi p'\|_{\mathcal{P}^0} + \|\mathrm{tr}\chi p'\|_{L^2(\mathcal{H})} + \|\nabla\!\!\!\!/_L p'\|_{L_t^2 L_x^2}\\
&\quad\quad+ \sum_{k>0}\left(\left\|E_k r^{-1}\int_0^t {r'}^2 \overline{an\mathrm{tr}\chi}\mathrm{tr}\chi p' dt'\right\|_{L_t^2 L_\omega^2}\right.\\
&\quad\quad\quad\left.+\left\|E_k r^{-1}\int_0^t {r'}^2 an\left((\mathrm{tr}\chi)^2 + |\hat{\chi}|^2\right) p' dt'\right\|_{L_t^2 L_\omega^2}\right).
\end{aligned}$$

Using (5.2), (4.21), BA1 and (2.79),

$$\begin{aligned}
\sum_{k>0}\left\|E_k r^{-1}\int_0^t {r'}^2 an|\hat{\chi}|^2\cdot p' dt'\right\|_{L_t^\infty L_\omega^2} &\lesssim \left(\mathcal{N}_1(\hat{\chi}) + \|\hat{\chi}\|_{L_\omega^\infty L_t^2}\right)\|an\hat{\chi}\cdot p'\|_{\mathcal{P}^0}\\
&\lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(p') \lesssim \Delta_0^2 + \mathcal{R}_0,
\end{aligned}$$

where for the last inequality, we employed (7.18).

It is easy to see by (4.20)

$$\sum_{k>0}\left\|E_k r^{-1}\int_0^t {r'}^2 \overline{an\mathrm{tr}\chi}\mathrm{tr}\chi p' dt'\right\|_{L_t^2 L_\omega^2} \lesssim \|\mathrm{tr}\chi p'\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(p').$$

and

$$\begin{aligned}
&\sum_{k>0}\left\|E_k r^{-1}\int_0^t {r'}^2 an(\mathrm{tr}\chi)^2 p' dt'\right\|_{L_t^2 L_\omega^2}\\
&\quad\le \sum_{k>0}\left(\left\|E_k r^{-1}\int_0^t {r'}^2 an\kappa\mathrm{tr}\chi p' dt'\right\|_{L_t^2 L_\omega^2} + \left\|E_k r^{-1}\int_0^t {r'}^2\overline{an\mathrm{tr}\chi}\mathrm{tr}\chi p' dt'\right\|_{L_t^2 L_\omega^2}\right)\\
&\quad\lesssim \|an\kappa\cdot r\mathrm{tr}\chi p'\|_{\mathcal{P}^0} + \|\mathrm{tr}\chi p'\|_{\mathcal{P}^0} \lesssim (\Delta_0^2 + \mathcal{R}_0 + 1)\mathcal{N}_1(p')
\end{aligned} \tag{7.22}$$

where we employed (4.20) and

$$\|an\kappa\cdot\mathrm{tr}\chi p'\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(p')(\Delta_0^2 + \mathcal{R}_0). \tag{7.23}$$

To see (7.23), we first deduce with the help of (4.13) that

$$\|an\kappa\cdot r\mathrm{tr}\chi p'\|_{\mathcal{P}^0} \lesssim \|\kappa\cdot r\mathrm{tr}\chi p'\|_{\mathcal{P}^0}. \tag{7.24}$$

By (4.17) with $G = r\mathrm{tr}\chi\kappa$

$$\|\kappa\cdot r\mathrm{tr}\chi p'\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(p')\left(\|r^{\frac{1}{2}}\nabla\!\!\!\!/(r\mathrm{tr}\chi\kappa)\|_{L^2(\mathcal{H})} + \|r^{1-\frac{1}{b}}\mathrm{tr}\chi\kappa\|_{L_t^b L_x^2}\right) \tag{7.25}$$

where $b > 4$. Since $\|r\kappa\|_{L^\infty} \lesssim C$ and

$$\begin{aligned}\|r^{\frac{1}{2}}\nabla\!\!\!/(r\mathrm{tr}\chi\kappa)\|_{L^2(\mathcal{H})} &\lesssim \|r^{\frac{3}{2}}\nabla\!\!\!/\mathrm{tr}\chi\kappa\|_{L^2(\mathcal{H})} + \|r^{\frac{3}{2}}\mathrm{tr}\chi\nabla\!\!\!/\kappa\|_{L^2(\mathcal{H})}\\ &\lesssim \|\nabla\!\!\!/\mathrm{tr}\chi\|_{L^2(\mathcal{H})} + \|\nabla\!\!\!/\kappa\|_{L^2(\mathcal{H})}\\ &\lesssim \Delta_0^2 + \mathcal{R}_0\end{aligned}$$

where for the last inequality, we employed (2.79) and Proposition 2.7. And by (2.38), we have

$$\|r^{1-\frac{1}{b}}\mathrm{tr}\chi\kappa\|_{L_t^b L_x^2} \lesssim \|r^{-\frac{1}{b}}\kappa\|_{L_t^b L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Thus in view of (7.25) and (7.24), (7.23) is proved. In view of (7.21), (7.19) is proved. $\square$

Similar to (7.17), we can obtain

$$\|r^{\frac{1}{2}}\mu\|_{\mathcal{B}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

8. **Appendix**

8.1. **Dyadic sobolev inequalities.** We start with proving Lemma 6.3 and a few useful consequences, which will be used to prove (6.76) in the second subsection.

Let us still regard κ and ι as elements of A. Using Propositions 2.5, 2.7, Lemma 2.8 and (1.13), more precisely

$$\|\underline{K}\|_{L_t^2 L_x^2} + \|\beta\|_{L_t^2 L_x^2} + \mathcal{N}_1(A) \lesssim \Delta_0^2 + \mathcal{R}_0,$$

we can adapt the approach in [4, Lemma 5.3] and [10, Chapter 9] to derive

Lemma 8.1. *For any smooth S_t tangent tensor fields F and all $q < 2$ sufficiently close to $q = 2$,*

$$\|r^{\frac{1}{2}-\frac{1}{q}}[P_k, \mathcal{D}_t]F\|_{L_t^q L_x^2} + 2^{-k}\|r^{\frac{3}{2}-\frac{1}{q}}\nabla\!\!\!/[P_k, \mathcal{D}_t]F\|_{L_t^q L_x^2} \lesssim 2^{-\frac{k}{2}+}\mathcal{N}_1(F), \tag{8.1}$$

$$\|r^{-\frac{1}{2}}[P_k, \mathcal{D}_t]F\|_{L_t^1 L_x^2} + 2^{-k}\|r^{\frac{1}{2}}\nabla\!\!\!/[P_k, \mathcal{D}_t]F\|_{L_t^1 L_x^2} \lesssim 2^{-k+}\mathcal{N}_1(F). \tag{8.2}$$

Now we are ready to prove Lemma 6.3.

Proof of Lemma 6.3. The result is trivial when $q = 2$. So we only need to consider the case $q > 2$. It is easy to get the following estimate

$$\|r^{-\frac{1}{2}-\frac{1}{q}}P_k F\|_{L_t^q L_x^2} \lesssim \|r^{-1}P_k F\|_{L_t^2 L_x^2}^{\frac{2}{q}} \|r^{-\frac{1}{2}}P_k F\|_{L_t^\infty L_x^2}^{\frac{q-2}{q}}.$$

Moreover we have by integrating along an arbitrary null geodesic,

$$\begin{aligned}\|r^{-\frac{1}{2}}P_k F\|_{L_x^2 L_t^\infty}^2 &\lesssim \|P_k(\mathcal{D}_t F)\|_{L_t^2 L_x^2}\|r^{-1}P_k F\|_{L_t^2 L_x^2} + \|r^{-1}[P_k, \mathcal{D}_t]F \cdot P_k F\|_{L_t^1 L_x^1}\\ &\quad + \|r^{-1}P_k F\|_{L_t^2 L_x^2}^2.\end{aligned} \tag{8.3}$$

We can obtain the following estimate with $\frac{1}{q'} + \frac{1}{q} = 1$ and $2 < q < \infty$

$$\begin{aligned}\|r^{-1}[P_k, \mathcal{D}_t F] \cdot P_k F\|_{L_t^1 L_x^1} &\lesssim \|r^{\frac{1}{2}-\frac{1}{q}}[P_k, \mathcal{D}_t]F\|_{L_t^{q'} L_x^2}\|r^{-\frac{1}{q}-\frac{1}{2}}P_k F\|_{L_t^q L_x^2}\\ &\lesssim 2^{-\frac{k}{2}+}\mathcal{N}_1(F)\|r^{-\frac{1}{q}-\frac{1}{2}}P_k F\|_{L_t^q L_x^2}\end{aligned} \tag{8.4}$$

where we employed Lemma 8.1 to derive the last inequality.

Combining the above estimates we obtain

$$\|r^{-\frac12-\frac1q}P_kF\|^q_{L^q_tL^2_x}\lesssim\|r^{-1}P_kF\|^2_{L^2_tL^2_x}\Big(\|P_k(\mathcal{D}_tF)\|_{L^2_tL^2_x}\|r^{-1}P_kF\|_{L^2_tL^2_x}$$
$$+2^{-\frac{k}{2}+}\mathcal{N}_1(F)\|r^{-\frac1q-\frac12}P_kF\|_{L^q_tL^2_x}+\|r^{-1}P_kF\|^2_{L^2_tL^2_x}\Big)^{\frac q2-1} \tag{8.5}$$

Using (4.4), we get for any $0\le\alpha<\frac{\frac12-\frac1q}{\frac q2+1}$,

$$\|r^{-\frac12-\frac1q}P_kF\|_{L^q_tL^2_x}\lesssim 2^{-k(\frac12+\frac1q)}(1+2^{-\alpha k})\mathcal{N}_1(F).$$

Combine (8.3), (8.4) and (6.32) with $2<q<\infty$, by using (4.4) we obtain

$$\|r^{-\frac12}P_kF\|_{L^\infty_tL^2_x}\lesssim 2^{-\frac12 k}\mathcal{N}_1(F).$$

To see (6.33), we derive by (**Sob**) and (4.3)

$$\|r^{-\frac1q}F_k\|_{L^q_tL^4_x}\lesssim\|r^{\frac12-\frac1q}\nabla\!\!\!\!/ F_k\|^{\frac12}_{L^q_tL^2_x}\|r^{-\frac12-\frac1q}F_k\|^{\frac12}_{L^q_tL^2_x}+\|r^{-\frac12-\frac1q}F_k\|_{L^q_tL^2_x}$$
$$\lesssim(2^{\frac k2}+1)\|r^{-\frac12-\frac1q}P_kF\|_{L^q_tL^2_x}.$$

Combined with (6.32), (6.33) follows. □

Lemma 8.2. *Let $\mathcal{D}^{-1}$ denote one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$ and ${}^\star\mathcal{D}_1^{-1}$. There hold the following estimates for appropriate S tangent tensor G,*

$$\|\mathcal{D}^{-1}P_k^2G\|_{L^\infty_tL^2_x}\lesssim r^{\frac32}2^{-\frac{3k}{2}}\mathcal{N}_1(G),\quad k>0, \tag{8.6}$$

$$\|\nabla\!\!\!\!/\mathcal{D}^{-1}P_k^2G\|_{L^\infty_tL^4_x}\lesssim\mathcal{N}_1(G)\left(1+2^{-\frac k2}r^{\frac12}\|\underline{K}\|^{\frac12}_{L^2_x}\right),\quad k>0. \tag{8.7}$$

Proof. (8.6) can be obtained by using Lemma 4.3 and (6.32). Now consider (8.7). For any S tangent tensor fields F, define

$$\|F\|_{H^1_x}=\|\nabla\!\!\!\!/ F\|_{L^2(S)}+\|r^{-1}F\|_{L^2(S)}.$$

Recall by Böchner identity contained in [3], there holds

$$\|\nabla\!\!\!\!/^2F\|_{L^2_x}\lesssim\|\Delta\!\!\!\!/ F\|_{L^2_x}+\|\underline{K}\cdot F\|_{L^2_x}+\|\underline{K}\|^{\frac12}_{L^2_x}\|\nabla\!\!\!\!/ F\|_{L^4_x}+r^{-1}\|F\|_{H^1_x}, \tag{8.8}$$

and by Sobolev embedding

$$\|F\|_{L^\infty_x}\lesssim r^{\frac1p}\|\nabla\!\!\!\!/^2F\|^{\frac1p}_{L^2_x}\|F\|^{\frac{p-1}{p}}_{H^1_x}+\|F\|_{H^1_x},\quad 2<p<\infty. \tag{8.9}$$

It is easy to observe from [1, P.38] that symbolically

$$\Delta\!\!\!\!/={}^\star\mathcal{D}\mathcal{D}\pm(\underline{K}+r^{-2}Id). \tag{8.10}$$

Hence with $2<p<\infty$,
(8.11)

$$\|\nabla\!\!\!\!/^2F\|_{L^2_x}\lesssim\|{}^\star\mathcal{D}\mathcal{D}F\|_{L^2_x}+\|\underline{K}\|^{\frac{p}{p-1}}_{L^2_x}r^{\frac{1}{p-1}}\|F\|_{H^1_x}+\|\underline{K}\|_{L^2_x}\|\nabla\!\!\!\!/ F\|_{L^2_x}+r^{-1}\|F\|_{H^1_x}.$$

For $F=\mathcal{D}^{-1}H$, using Proposition 3.4

$$\|\nabla\!\!\!\!/^2\mathcal{D}^{-1}H\|_{L^2_x}\lesssim\|{}^\star\mathcal{D}H\|_{L^2_x}+\|\underline{K}\|^{\frac{p}{p-1}}_{L^2_x}r^{\frac{1}{p-1}}\|\mathcal{D}^{-1}H\|_{H^1_x}+\|\underline{K}\|_{L^2_x}\|\nabla\!\!\!\!/\mathcal{D}^{-1}H\|_{L^2_x}$$
$$+r^{-1}\|\mathcal{D}^{-1}H\|_{H^1_x}$$
$$\lesssim\|{}^\star\mathcal{D}H\|_{L^2_x}+\|\underline{K}\|^{\frac{p}{p-1}}_{L^2_x}r^{\frac{1}{p-1}}\|H\|_{L^2_x}+\left(\|\underline{K}\|_{L^2_x}+r^{-1}\right)\|H\|_{L^2_x}$$

Let $H = P_k^2 G$, by (4.3), Proposition 3.4, Lemma 4.3, we obtain,

$$\begin{aligned}\|\nabla\!\!\!/^2\mathcal{D}^{-1}P_k^2G\|_{L_x^2} &\lesssim \|^\star\mathcal{D}P_k^2G\|_{L_x^2} + (\|\underline{K}\|_{L_x^2} + r^{-1})\|P_k^2G\|_{L_x^2} \\ &\lesssim \left((2^k+1)r^{-1} + \|\underline{K}\|_{L_x^2}\right)\|P_kG\|_{L_x^2}.\end{aligned}$$

By (**Sob**), Proposition 3.4, (6.32)

$$\begin{aligned}\|\nabla\!\!\!/\mathcal{D}^{-1}P_k^2G\|_{L_x^4} &\lesssim \|\nabla\!\!\!/^2\mathcal{D}^{-1}P_k^2G\|_{L_x^2}^{\frac12}\|\nabla\!\!\!/\mathcal{D}^{-1}P_k^2G\|_{L_x^2}^{\frac12} + r^{-\frac12}\|\nabla\!\!\!/\mathcal{D}^{-1}P_k^2G\|_{L_x^2} \\ &\lesssim (1 + 2^{-\frac{k}{2}}r^{\frac12}\|\underline{K}\|_{L_x^2}^{\frac12})\mathcal{N}_1(G).\end{aligned}$$

□

8.2. **Proof of (6.76).** We first prove (6.76) by assuming the following results.

Lemma 8.3. *Denote by $\mathcal{D}^{-1}G$ one of the terms $\mathcal{D}_1^{-1}G$, $\mathcal{D}_2^{-1}G$ and $^\star\mathcal{D}_1^{-1}G$ for appropriate S tangent tensor G. Let $F = \mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G$, we have*

$$\|\nabla\!\!\!/F\|_{B_{2,1}^0} \lesssim \mathcal{N}_1(G)\left(\Delta_0^2 + \mathcal{R}_0 + c_0 r^{\frac12}(\|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} + \|\check{R}\|_{L_x^2} + r^2\|\mathcal{D}\check{R}\|_{L_x^\infty})\right)$$

where c_0 depends on $\|r\underline{K}\|_{L_x^\infty} + \underline{K}_{\alpha_0}$.

Lemma 8.4. *Let $\mathcal{D}^{-1}$ denotes one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$ and $^\star\mathcal{D}_1^{-1}$. For S tangent tensor H, there hold*

$$\|\nabla\!\!\!/\mathcal{D}^{-1}H\|_{B_{2,1}^1} \lesssim \|^\star\mathcal{D}H\|_{B_{2,1}^0} + \|H\|_{L_\omega^2} + c_0 r^{\frac12}\mathcal{N}_1(H), \tag{8.12}$$

$$\|\nabla\!\!\!/\mathcal{D}^{-1}H\|_{L_x^\infty} \le \|^\star\mathcal{D}H\|_{B_{2,1}^0} + (c_0 r^\theta + 1)(\|\nabla\!\!\!/H\|_{L_x^2} + \|H\|_{L_\omega^2} + c_0 r^{\frac12}\mathcal{N}_1(H)). \tag{8.13}$$

where c_0 is depending on the quantity $r\|\underline{K}\|_{L_x^\infty} + \underline{K}_{\alpha_0} + r\|\nabla\!\!\!/\underline{K}\|_{L_x^2}$, and $\theta > 0$ is very close to 0.

Let us set $P = \sum_{i=1}^\infty P_i$ and $\tilde{P} = \nabla\!\!\!/\mathcal{D}^{-1}P$. Since $\bar{P} = \tilde{P} + P - P_1$ and in view of (6.73) that $\lim_{t\to0}\|P_1\|_{L_\omega^\infty} = 0$, (6.76) can be proved by establishing

$$\lim_{t\to0}\|P\|_{L_x^\infty} = 0, \quad \text{and} \quad \lim_{t\to0}\|\tilde{P}\|_{L_x^\infty} < \infty. \tag{8.14}$$

In view of (6.86), we have by (**SobM2**), Lemma 3.1 and (6.89),

$$\begin{aligned}\|P_i\|_{L_\omega^\infty} &\le \|\mathcal{D}_1^{-1}\check{R}\|_{L_\omega^\infty}\|\mathcal{D}^{-1}P_{i-1}\|_{L_\omega^\infty} \lesssim r^{\frac12}\|\mathcal{D}_1^{-1}\check{R}\|_{L_\omega^\infty}\mathcal{N}_2(\mathcal{D}^{-1}P_{i-1}) \\ &\lesssim r^{\frac12}\mathcal{N}_1(P_{i-1})\|\mathcal{D}_1^{-1}\check{R}\|_{L_\omega^\infty} \lesssim r^{\frac12}(C(\Delta_0^2 + \mathcal{R}_0))^{i-1}\mathcal{N}_2(F)\|\mathcal{D}_1^{-1}\check{R}\|_{L_\omega^\infty}.\end{aligned}$$

Summing over $i \ge 1$, with $(\Delta_0^2 + \mathcal{R}_0)$ sufficiently small,

$$\|P\|_{L_\omega^\infty} \lesssim r^{\frac12}\mathcal{N}_2(F)\|\mathcal{D}_1^{-1}\check{R}\|_{L_\omega^\infty}.$$

Noting that $\lim_{t\to0}\|\mathcal{D}_1^{-1}\check{R}\|_{L_\omega^\infty} < \infty$ and $\mathcal{N}_2(F) < \infty$,

$$\lim_{t\to0}\|P\|_{L_x^\infty} = 0.$$

It remains to prove the second part of (8.14). By (8.13), there holds

$$\|\tilde{P}\|_{L_x^\infty} \le \|\nabla\!\!\!/P\|_{B_{2,1}^0} + (c_0 r^\theta + 1)\left(\|\nabla\!\!\!/P\|_{L_x^2} + \|P\|_{L_\omega^2} + c_0 r^{\frac12}\mathcal{N}_1(P)\right). \tag{8.15}$$

Recall the definition of P_i in (6.86). For each $P_i = \mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}P_{i-1}$, by Lemma 8.3, there holds

$$\begin{aligned}\|\nabla\!\!\!/ P_i\|_{B^0_{2,1}} &\lesssim c_0\mathcal{N}_1(P_{i-1})r^{\frac12}\left(\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}+\|\check{R}\|_{L^2_x}+r^2\|\mathcal{D}\check{R}\|_{L^\infty_x}\right)\\&\quad+\mathcal{N}_1(P_{i-1})(\Delta_0^2+\mathcal{R}_0).\end{aligned}$$

In view of (6.89), summing over $i\ge 1$ gives

$$\begin{aligned}\|\nabla\!\!\!/ P\|_{B^0_{2,1}} &\lesssim \sum_{i\ge1}\|\nabla\!\!\!/ P_i\|_{B^0_{2,1}}\\&\lesssim c_0\sum_{i\ge1}\left(C(\Delta_0^2+\mathcal{R}_0)\right)^i\mathcal{N}_2(F)r^{\frac12}\left(\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}+\|\check{R}\|_{L^2_x}+r^2\|\mathcal{D}\check{R}\|_{L^\infty_x}\right)\\&\quad+\sum_{i\ge1}(C(\Delta_0^2+\mathcal{R}_0))^i\mathcal{N}_2(F)(\Delta_0^2+\mathcal{R}_0).\end{aligned}$$

Hence

$$\begin{aligned}\|\nabla\!\!\!/ P\|_{B^0_{2,1}} &\lesssim c_0\mathcal{N}_2(F)r^{\frac12}\left(\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}+\|\check{R}\|_{L^2_x}+r^2\|\mathcal{D}\check{R}\|_{L^\infty_x}\right)\\&\quad+\mathcal{N}_2(F)(\Delta_0^2+\mathcal{R}_0).\end{aligned}\tag{8.16}$$

By (6.86), (**SobM1**), Lemma 3.1, (6.89) and (6.22),

$$\begin{aligned}\|P_i\|_{L^2_\omega} &\lesssim r^{-1}\|\mathcal{D}_1^{-1}\check{R}\|_{L^4_x}\|\mathcal{D}^{-1}P_{i-1}\|_{L^4_x}\lesssim\mathcal{N}_1(\mathcal{D}^{-1}\check{R})r^{-1}\mathcal{N}_1(\mathcal{D}^{-1}P_{i-1})\\&\lesssim\mathcal{N}_1(P_{i-1})(\Delta_0^2+\mathcal{R}_0)\lesssim(C(\Delta_0^2+\mathcal{R}_0))^{i-1}(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F).\end{aligned}$$

Therefore

$$\|P\|_{L^2_\omega}\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F).\tag{8.17}$$

It is easy to derive from $P=\sum_{i\ge1}P_i$ and (6.89) that

$$\mathcal{N}_1(P)\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F).\tag{8.18}$$

Thus in view of (8.15), the combination of (8.16), (8.17) and (8.18) implies

$$\lim_{t\to0}\|\tilde{P}\|_{L^\infty_x}\lesssim\mathcal{N}_2(F)(\Delta_0^2+\mathcal{R}_0)<\infty,$$

as desired.

Proof of Lemma 8.3. Let us compute $\|\nabla\!\!\!/ F\|_{L^2_x}$ first.

$$\begin{aligned}\|\nabla\!\!\!/ F\|_{L^2_x}&=\|\nabla\!\!\!/(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G)\|_{L^2_x}\\&\lesssim\|\nabla\!\!\!/\mathcal{D}^{-1}\check{R}\|_{L^2_x}\|\mathcal{D}^{-1}G\|_{L^\infty_x}+\|\mathcal{D}^{-1}\check{R}\|_{L^4_x}\|\nabla\!\!\!/ D^{-1}G\|_{L^4_x}.\end{aligned}$$

By (**SobM2**), (**SobM1**), Proposition 3.4, Lemma 3.1 and (6.22),

$$\begin{aligned}\|\nabla\!\!\!/ F\|_{L^2_x}&\lesssim r^{\frac12}\|\check{R}\|_{L^2_x}\mathcal{N}_2(\mathcal{D}^{-1}G)+\mathcal{N}_1(\mathcal{D}^{-1}\check{R})\mathcal{N}_1(\nabla\!\!\!/\mathcal{D}^{-1}G)\\&\lesssim\left(r^{\frac12}\|\check{R}\|_{L^2_x}+\mathcal{N}_1(\mathcal{D}^{-1}\check{R})\right)\mathcal{N}_1(G)\\&\lesssim\left(r^{\frac12}\|\check{R}\|_{L^2_x}+\Delta_0^2+\mathcal{R}_0\right)\mathcal{N}_1(G).\end{aligned}$$

Now we prove

$$\sum_{k>0}\|P_k\nabla\!\!\!/(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G)\|_{L^2_x}\lesssim c_0\mathcal{N}_1(G)r^{\frac12}\left(\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}+\|\check{R}\|_{L^2_x}+r^2\|\nabla\!\!\!/\check{R}\|_{L^\infty_x}\right).\tag{8.19}$$

Indeed, we will employ GLP decompositions to write

$$G = \sum_{m>0} P_m^2 G + P_{\le 0} G + U(\infty) G.$$

For simplicity, we consider the high frequency terms $\sum_{m>0} P_m^2 G$. The other two terms can be treated similar to *Case 2*.

Case 1: $k < m$. By (4.3) and (8.6),

$$\|P_k \nabla\!\!\!/ (\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m)\|_{L_x^2} \lesssim 2^{k-\frac{3m}{2}} r^{\frac{1}{2}} \mathcal{N}_1(G) \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty}.$$

Thus we obtain

$$\sum_{k>0}\sum_{m>k} \|P_k \nabla\!\!\!/ (\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m)\|_{L_x^2} \lesssim r^{\frac{1}{2}} \mathcal{N}_1(G) \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty}.$$

Case 2: $k > m$. We decompose further such that

$$P_k \nabla\!\!\!/ (\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m) = P_k \nabla\!\!\!/ (\sum_{n>0} P_n^2 + P_{\le 0})(\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m). \tag{8.20}$$

For simplicity we consider the high frequency terms, and the low frequency terms can be treated similarly. We can adapt the proof for [11, Proposition 4.5] to obtain the following inequality for S tangent tensor field F and $1 > \alpha > \alpha_0 \ge \frac{1}{2}$

$$\begin{aligned} \|P_k \nabla\!\!\!/ P_n^2 F\|_{L_x^2} \lesssim \Big(& 2^{\min(k,n)} 2^{-2|n-k|} r^{-1} + 2^{\min(k,n)} 2^{-(1-\alpha)\max(k,n)} \underline{K}_{\alpha_0} r^{-\alpha} \\ & + 2^{-|k-n|} \|\underline{K}\|_{L_x^2}^{\alpha} \underline{K}_{\alpha_0} \Big) \|P_n F\|_{L_x^2}. \end{aligned} \tag{8.21}$$

Let $\mathcal{I}_{nm} = \|P_n(\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m)\|_{L_x^2}$, we have

$$\begin{aligned} & \|P_k \nabla\!\!\!/ P_n^2 (\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m)\|_{L_x^2} \\ & \quad \lesssim \Big(2^{\min(k,n)} 2^{-2|n-k|} r^{-1} + 2^{\min(k,n)} 2^{-(1-\alpha)\max(k,n)} \underline{K}_{\alpha_0} r^{-\alpha} \\ & \quad + 2^{-|k-n|} \|\underline{K}\|_{L_x^2}^{\alpha} \underline{K}_{\alpha_0} \Big) \mathcal{I}_{nm}. \end{aligned} \tag{8.22}$$

Now we estimate $\mathcal{I}_{nm}$. Let us first consider the case that $n > m > 0$. By Proposition 4.1 (iii), we have

$$\begin{aligned} \mathcal{I}_{nm} & \lesssim r^2 2^{-2n} \|\tilde{P}_n \Delta\!\!\!/ (\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m)\|_{L_x^2} \\ & \lesssim r^2 2^{-2n} \Big(\|\Delta\!\!\!/ \mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m\|_{L_x^2} + \|\tilde{P}_n (\nabla\!\!\!/ \mathcal{D}^{-1}\check{R} \cdot \nabla\!\!\!/ \mathcal{D}^{-1} G_m)\|_{L_x^2} \\ & \quad + \|\Delta\!\!\!/ \mathcal{D}^{-1} G_m \cdot \mathcal{D}^{-1}\check{R}\|_{L_x^2} \Big). \end{aligned}$$

By (8.6) and (8.10), we have

$$\begin{aligned} & \|\Delta\!\!\!/ \mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m\|_{L_x^2} \\ & \quad \lesssim \left(\|\nabla\!\!\!/ \check{R}\|_{L_x^\infty} + \|\underline{K}\|_{L_x^\infty} \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} + r^{-2} \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} \right) 2^{-\frac{3m}{2}} r^{\frac{3}{2}} \mathcal{N}_1(G). \end{aligned}$$

By (4.5), Propositions 3.4 and (8.7),

$$\begin{aligned} \|\tilde{P}_n (\nabla\!\!\!/ \mathcal{D}^{-1}\check{R} \cdot \nabla\!\!\!/ \mathcal{D}^{-1} G_m)\|_{L_x^2} & \lesssim 2^{\frac{n}{2}} r^{-\frac{1}{2}} \|\nabla\!\!\!/ \mathcal{D}^{-1}\check{R}\|_{L_x^2} \|\nabla\!\!\!/ \mathcal{D}^{-1} G_m\|_{L_x^4} \\ & \lesssim 2^{\frac{n}{2}} r^{-\frac{1}{2}} \|\check{R}\|_{L_x^2} \mathcal{N}_1(G) \left(1 + 2^{-\frac{m}{2}} r^{\frac{1}{2}} \|\underline{K}\|_{L_x^2} \right). \end{aligned}$$

By (8.10), (4.3) and (6.32), (8.6), we obtain

$$\begin{aligned}&\|\not{\Delta}\mathcal{D}^{-1}G_m\cdot\mathcal{D}^{-1}\check{R}\|_{L^2_x}\\&\qquad\lesssim\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}\left(\|{}^\star\mathcal{D}G_m\|_{L^2_x}+\|\underline{K}\|_{L^\infty_x}\|\mathcal{D}^{-1}G_m\|_{L^2_x}+r^{-2}\|\mathcal{D}^{-1}G_m\|_{L^2_x}\right)\\&\qquad\lesssim\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}\left(2^{\frac{m}{2}}r^{-\frac12}+\|\underline{K}\|_{L^\infty_x}2^{-\frac{3m}{2}}r^{\frac32}+r^{-\frac12}2^{-\frac{3m}{2}}\right)\mathcal{N}_1(G).\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{I}_{nm}\lesssim r^22^{-2n}\mathcal{N}_1(G)\Big(&2^{\frac{m}{2}}r^{-\frac12}\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}+2^{\frac{n}{2}}r^{-\frac12}\|\check{R}\|_{L^2_x}\left(1+2^{-\frac{m}{2}}r^{\frac12}\|\underline{K}\|_{L^2_x}\right)\\&+\|\not\nabla\check{R}\|_{L^\infty_x}2^{-\frac{3m}{2}}r^{\frac32}+\|\underline{K}\|_{L^\infty_x}\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}2^{-\frac{3m}{2}}r^{\frac32}\Big).\end{aligned}\tag{8.23}$$

Combined with (8.22), summing over $k,n,m>0$ for the cases $k>n>m$ and $n>k>m$, we summarize the results as follows

$$\begin{aligned}\sum_{k,n,m>0,k>m,n>m}&\|P_k\not\nabla P_n^2(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G_m)\|_{L^2_x}\\&\lesssim c_0\mathcal{N}_1(G)r^{\frac12}\left(\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}+\|\check{R}\|_{L^2_x}+r^2\|\not\nabla\check{R}\|_{L^\infty_x}\right)\end{aligned}$$

and c_0 depends on $\|r^2\underline{K}\|_{L^\infty_x}+\underline{K}_{\alpha_0}+\|\underline{K}\|_{L^2_x}$.

It remains to estimate $\mathcal{I}_{nm}$ when $k>m>n$. By (8.6),

$$\mathcal{I}_{nm}\lesssim\|\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G_m\|_{L^2_x}\lesssim 2^{-\frac{3m}{2}}r^{\frac32}\mathcal{N}_1(G)\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}.\tag{8.24}$$

Combined with (8.21)

$$\sum_{k,n,m>0,k>m>n}\|P_k\not\nabla P_n^2(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G_m)\|_{L^2_x}\le c_0\mathcal{N}_1(G)r^{\frac12}\|\mathcal{D}^{-1}\check{R}\|_{L^\infty_x}\tag{8.25}$$

□

Recall the following expression holds symbolically for any S tangent tensor F, (see [3],[11, page 300]),

$$[\not\nabla,\not\Delta]F=\not\nabla(K\cdot F)+K\cdot\not\nabla F.\tag{8.26}$$

Proof of Lemma 8.4. Assuming (8.12), we first prove (8.13). For simplicity let us set $\tilde{H}=\not\nabla\mathcal{D}^{-1}H$. We have by [4, Proposition 3.20 (x)],

$$\|\tilde{H}\|_{L^\infty_x}\lesssim\sum_{k>0}2^kr^{-1}\|P_k\tilde{H}\|_{L^2_x}+r^\theta c_0(\|\not\nabla\tilde{H}\|_{L^2_x}+\|r^{-1}\tilde{H}\|_{L^2_x}),\tag{8.27}$$

where c_0 depends on $\|\underline{K}\|_{L^2_x}$, and $\theta>0$ is very close to 0.

The first term on the right of (8.27) can be bounded in view of (8.12),

$$\|\tilde{H}\|_{B^1_{2,1}}\lesssim\|{}^\star\mathcal{D}H\|_{B^0_{2,1}}+\|H\|_{L^2_\omega}+r^{\frac12}c_0\mathcal{N}_1(H).\tag{8.28}$$

We then estimate $\|\not\nabla\tilde{H}\|_{L^2_x}$, by applying (8.11) to $F=\mathcal{D}^{-1}H$. By using Proposition 3.4 and (**SobM1**) we can obtain

$$\|\not\nabla\tilde{H}\|_{L^2_x}\lesssim\|{}^\star\mathcal{D}H\|_{L^2_x}+r^{-1}\|H\|_{L^2_x}+r^{\frac12}c_0\mathcal{N}_1(H),\tag{8.29}$$

with c_0 depending on $\|\underline{K}\|_{L^2_x}$.

By combining (8.29), (8.28) and (8.27) and Proposition 3.4, (8.13) follows.

Now consider (8.12). Using GLP projections, we need to prove

$$\sum_{k,m>0} 2^k r^{-1}\|P_k \nabla\!\!\!/ P_m^2 \mathcal{D}^{-1} H\|_{L_x^2} \lesssim \|{}^\star\mathcal{D}H\|_{B^0_{2,1}} + c_0 r^{\frac{1}{2}}\mathcal{N}_1(H) \tag{8.30}$$

$$\sum_{k,m>0} 2^k r^{-1}\|P_k \nabla\!\!\!/ P_{\le 0} \mathcal{D}^{-1} H\|_{L_x^2} \lesssim c_0 r^{\frac{1}{2}}\mathcal{N}_1(H), \tag{8.31}$$

where c_0 depends on $\|\underline{K}\|_{L_x^2} + r\|\nabla\!\!\!/ \underline{K}\|_{L_x^2} + r\|\underline{K}\|_{L_x^\infty}$.

The proof of (8.31) is similar to the following *Case 1* of the treatment for

$$\mathcal{I}_{km} := 2^k r^{-1}\|P_k \nabla\!\!\!/ P_m^2 \mathcal{D}^{-1} H\|_{L_x^2},$$

thus we will give the proof of (8.30) only.

Case 1: $k > m$. By (4.2),

$$\mathcal{I}_{km} \le 2^{-k} r\left(\|P_k \nabla\!\!\!/ \Delta\!\!\!/ P_m^2 \mathcal{D}^{-1} H\|_{L_x^2} + \|P_k[\Delta\!\!\!/, \nabla\!\!\!/] P_m^2 \mathcal{D}^{-1} H\|_{L_x^2}\right). \tag{8.32}$$

Let us denote the two terms on the right by $\mathcal{I}^1_{km}$ and $\mathcal{I}^2_{km}$ respectively. In view of (8.10), (4.3) and Lemma 4.3 we have

$$\begin{aligned}\mathcal{I}^1_{km} &\lesssim 2^{-k} r\|P_k \nabla\!\!\!/ P_m^2({}^\star\mathcal{D}H + \underline{K}\mathcal{D}^{-1}H + r^{-2}\mathcal{D}^{-1}H)\|_{L_x^2}\\ &\lesssim 2^{m-k}\|P_m {}^\star\mathcal{D}H\|_{L_x^2} + 2^{-k} r\|H\|_{L_x^2} + 2^{-k} r\|P_k \nabla\!\!\!/ P_m^2(\underline{K}\cdot\mathcal{D}^{-1}H)\|_{L_x^2}\end{aligned} \tag{8.33}$$

We only need to employ (8.21) to estimate the last term of (8.33).

$$\begin{aligned}2^{-k} r\|P_k \nabla\!\!\!/ P_m^2(\underline{K}\cdot\mathcal{D}^{-1}H)\|_{L_x^2} \lesssim \Big(&2^{-3|m-k|} + 2^{-|m-k|}2^{-(1-\alpha)k}\underline{K}_{\alpha_0} r^{1-\alpha}\\ &+ 2^{-k}2^{-|k-m|} r\|\underline{K}\|^{\alpha}_{L_x^2}\underline{K}_{\alpha_0}\Big)\|P_m(\underline{K}\mathcal{D}^{-1}H)\|_{L_x^2}.\end{aligned} \tag{8.34}$$

For the last two terms, in view of $k > m$, (**SobM2**) and Lemma 3.1

$$\begin{aligned}\sum_{k>m}\Big(&2^{-|m-k|}2^{-(1-\alpha)k}\underline{K}_{\alpha_0} r^{1-\alpha} + 2^{-k}2^{-|k-m|} r\|\underline{K}\|^{\alpha}_{L_x^2}\underline{K}_{\alpha_0}\Big)\|P_m(\underline{K}\mathcal{D}^{-1}H)\|_{L_x^2}\\ &\lesssim \underline{K}_{\alpha_0}\left(r^{1-\alpha} + r\|\underline{K}\|^{\alpha}_{L_x^2}\right)\|\underline{K}\|_{L_x^2} r^{\frac{1}{2}}\mathcal{N}_1(H).\end{aligned} \tag{8.35}$$

Let us decompose $\underline{K} = \sum_n P_n^2\underline{K} + \bar{\underline{K}}$ and consider the high frequency term for the purpose of simplicity. With the help of Proposition 3.4, the proof contained in [11, pages 299–300] implies for $m, n > 0$

$$\|P_m(\underline{K}_n\mathcal{D}^{-1}H)\|_{L_x^2} \lesssim 2^{-\frac{3}{4}|m-n|}\|P_n\underline{K}\|_{L_x^2}\left(\|\mathcal{D}^{-1}H\|_{L_x^\infty} + \|H\|_{L_x^2}\right). \tag{8.36}$$

Therefore the first term on the right of (8.34) can be estimated as follows,

$$\begin{aligned}\sum_{k>m}\sum_{n>0}&2^{-3|m-k|}\|P_m(\underline{K}_n\mathcal{D}^{-1}H)\|_{L_x^2}\\ &\lesssim \sum_{k>m}\sum_{n>0}2^{-3|m-k|-\frac{3}{4}|m-n|}\|P_n\underline{K}\|_{L_x^2}\left(\|\mathcal{D}^{-1}H\|_{L_x^\infty} + \|H\|_{L_x^2}\right)\\ &\lesssim \|\underline{K}\|_{B^0_{2,1}} r^{\frac{1}{2}}\mathcal{N}_1(H)\end{aligned}$$

where we employed (**SobM2**), Lemma 3.1 and (**SobM1**) to obtain the last inequality. It is easy to check by (4.4) that $\|\underline{K}\|_{B^0_{2,1}} \lesssim \|\underline{K}\|_{L_x^2} + r\|\nabla\!\!\!/ \underline{K}\|_{L_x^2}$. Consequently,

$$\sum_{k>m}\mathcal{I}^1_{km} \lesssim \|{}^\star\mathcal{D}H\|_{B^0_{2,1}} + c_0 r^{\frac{1}{2}}\mathcal{N}_1(H).$$

Now we consider $\mathcal{I}^2_{km}$ with the help of (8.26) and (**SobM2**), also using (4.3) and Lemma 4.3

$$
\begin{aligned}
\mathcal{I}^2_{km} &\lesssim 2^{-k} r \left(\|P_k \slashed{\nabla}(K \cdot P^2_m \mathcal{D}^{-1} H)\|_{L^2_x} + \|P_k(K \cdot \slashed{\nabla} P^2_m \mathcal{D}^{-1} H)\|_{L^2_x} \right) \\
&\lesssim 2^{-k} r \Big(\|\slashed{\nabla} \underline{K}\|_{L^2_x} \|P^2_m \mathcal{D}^{-1} H\|_{L^\infty_x} + \|\underline{K}\|_{L^\infty_x} \|\slashed{\nabla} P^2_m \mathcal{D}^{-1} H\|_{L^2_x} \\
&\quad + r^{-2} \|P_k(\slashed{\nabla} P^2_m \mathcal{D}^{-1} H)\|_{L^2_x} \Big) \\
&\lesssim 2^{-k} \left(r \|\slashed{\nabla} \underline{K}\|_{L^2_x} r^{\frac{1}{2}} \mathcal{N}_2(\mathcal{D}^{-1} H) + r \|\underline{K}\|_{L^\infty_x} \|H\|_{L^2_x} + r^{-1} \|H\|_{L^2_x} \right).
\end{aligned}
$$

Also using Lemma 3.1 and (**SobM1**)

$$
\sum_{k>m>0} \mathcal{I}^2_{km} \lesssim \left(r \|\slashed{\nabla} \underline{K}\|_{L^2_x} + r \|\underline{K}\|_{L^\infty_x} \right) r^{\frac{1}{2}} \mathcal{N}_1(H) + r^{-1} \|H\|_{L^2_x}.
$$

Case 2: $k < m$. Consider $\mathcal{I}_{km}$ in this case by (4.2) and (4.1).

$$
\begin{aligned}
\mathcal{I}_{km} &\le 2^{k-2m} r \|P_k \slashed{\nabla} \slashed{\Delta} P^2_m \mathcal{D}^{-1} H\|_{L^2_x} \\
&\le 2^{k-2m} r \left(\|P_k \slashed{\Delta} \slashed{\nabla} P^2_m \mathcal{D}^{-1} H\|_{L^2_x} + \|P_k [\slashed{\nabla}, \slashed{\Delta}] P^2_m \mathcal{D}^{-1} H\|_{L^2_x} \right) \\
&\le 2^{3k-2m} r^{-1} \|P_k \slashed{\nabla} P^2_m \mathcal{D}^{-1} H\|_{L^2_x} + 2^{k-2m} r \|P_k [\slashed{\nabla}, \slashed{\Delta}] P^2_m \mathcal{D}^{-1} H\|_{L^2_x}. \qquad (8.37)
\end{aligned}
$$

Let us denote by $\mathcal{I}^1_{km}$ the first term in the line of (8.37) and by $\mathcal{I}^2_{km}$ the second term. Consider $\mathcal{I}^1_{km}$ first. By (4.3), (4.2) and (8.10)

$$
\begin{aligned}
\mathcal{I}^1_{km} &\lesssim 2^{4k-2m} r^{-2} \|P^2_m \mathcal{D}^{-1} H\|_{L^2_x} \lesssim 2^{4k-4m} \|P^2_m \slashed{\Delta} \mathcal{D}^{-1} H\|_{L^2_x} \\
&\lesssim 2^{-4|k-m|} \left(\|P^2_m {}^\star\mathcal{D} H\|_{L^2_x} + \|P^2_m (\underline{K} \mathcal{D}^{-1} H)\|_{L^2_x} + r^{-2} \|P^2_m \mathcal{D}^{-1} H\|_{L^2_x} \right)
\end{aligned}
$$

By (4.4), Proposition 3.4, (**SobM2**) and Lemma 3.1,

$$
\begin{aligned}
\|P^2_m (\underline{K} \mathcal{D}^{-1} H)\|_{L^2_x} &\lesssim 2^{-m} r \left(\|\slashed{\nabla} \underline{K}\|_{L^2_x} \|\mathcal{D}^{-1} H\|_{L^\infty_x} + \|\underline{K}\|_{L^\infty_x} \|H\|_{L^2_x} \right) \\
&\lesssim 2^{-m} r^{\frac{3}{2}} \mathcal{N}_1(H) \left(\|\slashed{\nabla} \underline{K}\|_{L^2_x} + \|\underline{K}\|_{L^\infty_x} \right). \qquad (8.38)
\end{aligned}
$$

Also using Lemma 4.3,

$$
\mathcal{I}^1_{km} \lesssim 2^{-4|k-m|} \left(\|P^2_m {}^\star\mathcal{D} H\|_{L^2_x} + r^{-1} 2^{-m} \|H\|_{L^2_x} + 2^{-m} r^{\frac{3}{2}} \mathcal{N}_1(H) (\|\slashed{\nabla} \underline{K}\|_{L^2_x} + \|\underline{K}\|_{L^\infty_x}) \right).
$$

Hence, we obtain

(8.39)

$$
\sum_{k,m>0,k<m} \mathcal{I}^1_{km} \lesssim \|{}^\star\mathcal{D} H\|_{B^0_{2,1}} + r^{-1} \|H\|_{L^2_x} + r^{\frac{1}{2}} \mathcal{N}_1(H) \left(r \|\slashed{\nabla} \underline{K}\|_{L^2_x} + r \|\underline{K}\|_{L^\infty_x} \right).
$$

Now consider $\mathcal{I}^2_{km}$ in view of (8.26).

$$
\begin{aligned}
\mathcal{I}^2_{km} &\lesssim 2^{k-2m} r \Big(\|P_k \slashed{\nabla} (\underline{K} P^2_m \mathcal{D}^{-1} H)\|_{L^2_x} + \|P_k (\underline{K} \slashed{\nabla} P^2_m \mathcal{D}^{-1} H)\|_{L^2_x} \\
&\quad + r^{-2} \|P_k \slashed{\nabla} P^2_m \mathcal{D}^{-1} H\|_{L^2_x} \Big).
\end{aligned}
$$

By (4.3) and Lemma 4.3

$$
\mathcal{I}^2_{km} \lesssim 2^{k-2m} r \left(\|\underline{K}\|_{L^\infty_x} \|H\|_{L^2_x} + 2^{k-m} r^{-2} \|H\|_{L^2_x} \right).
$$

Also using (**SobM1**),

$$
\sum_{k,m>0,m>k} \mathcal{I}^2_{km} \lesssim r^{-1} \|H\|_{L^2_x} + r^{\frac{1}{2}} \mathcal{N}_1(H) \cdot r \|\underline{K}\|_{L^\infty_x}.
$$

Thus

$$\sum_{k,m>0,m>k} \mathcal{I}_{km} \lesssim r^{-1}\|H\|_{L_x^2} + c_0 r^{\frac{1}{2}}\mathcal{N}_1(H)$$

where c_0 depends on $r(\|\underline{K}\|_{L_x^\infty} + \|\nabla\!\!\!/\,\underline{K}\|_{L_x^2})$. $\square$

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